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WEAK CONVERGENCE THEOREMS FOR NEARLY ASYMPTOTICALLY NONEXPANSIVE MAPPINGS AND ASYMPTOTICALLY NONEXPANSIVE NON-SELF MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Abstract. In this paper, we proposed a new two-step iteration scheme of hybrid mixed type for two nearly asymptotically nonexpansive self mappings and two asymptotically nonexpansive non-self mappings. Weak convergence theorems are established in uniformly convex Banach spaces. Our results extend and generalize the corresponding results given in the current existing literature.

Keywords. Asymptotically nonexpansive non-self mapping; Banach space; Common fixed point; Nearly asymptotically nonexpansive mapping; Weak convergence.

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1. Introduction and Preliminaries

Let K be a nonempty subset of a real Banach space E and let $T: K \to K$ be a nonlinear mapping. In this paper, we denote the set of all fixed points of T by $F(T)$. The set of common fixed points of four mappings S_1, S_2, T_1 and T_2 is denoted by $F = F(S_1) \bigcap F(S_2) \bigcap F(T_1) \bigcap F(T_2)$. Recall the following definitions.

T is said to be asymptotically nonexpansive [1] if there exists a positive sequence $\{k_n\}$ in $[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n(x) - T^n(y)\| \leq k_n\|x - y\|, \; \forall x, y \in K, \, n \in \mathbf{N}.$$

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T is said to be uniformly L-Lipschitzian if for some $L > 0$ such that

$$\|T^n(x) - T^n(y)\| \leq L\|x - y\|, \; \forall x, y \in K, n \in \mathbf{N}.$$

Also T is called a contraction if for some $0 < k < 1$ such that

$$\|T(x) - T(y)\| \leq k\|x - y\|, \; \forall x, y \in K.$$

Fix a sequence $\{e_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} e_n = 0$, then according to Agarwal *et al.* [2], T is said to be nearly asymptotically nonexpansive if $k_n \geq 1$ for all $n \in \mathbf{N}$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n(x) - T^n(y)\| \leq k_n(\|x - y\| + e_n), \; \forall x, y \in K.$$

T will be nearly uniformly L-Lipschitzian if $k_n \leq L$ for all $n \in \mathbf{N}$.

Remark 1.1. Every asymptotically nonexpansive mapping is nearly asymptotically nonexpansive and every nearly asymptotically nonexpansive mapping is nearly uniformly L-Lipschitzian.

Definition 1.1. *A subset K of a Banach space E is said to be a retract of E if there exists a continuous mapping $P\colon E \to K$ (called a retraction) such that $P(x) = x$ for all $x \in K$. If, in addition P is nonexpansive, then P is said to be a nonexpansive retract of E.*

If $P\colon E \to K$ is a retraction, then $P^2 = P$. A retract of a Hausdorff space must be a closed subset. Every closed convex subset of a uniformly convex Banach space is a retract.

Chidume *et al.* [3] introduced the concept of non-self asymptotically nonexpansive mappings as follows.

Definition 1.2. *Let K be a nonempty subset of a real Banach space E and let $P\colon E \to K$ be a nonexpansive retraction of E onto K. A non-self mapping $T: K \to E$ is said to be asymptotically nonexpansive if there exists a positive sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that*

$$\|T(PT)^{n-1}(x) - T(PT)^{n-1}(y)\| \leq k_n\|x - y\|, \; \forall x, y \in K, n \in \mathbf{N}.$$

Example 1.1. Let $E = \mathbb{R}$ be a normed linear space, $K = [0, 1]$ and P be the identity mapping. For each $x \in K$, we define

$$T(x) = \begin{cases} \lambda x, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where $0<\lambda<1$. Then $|T^n x - T^n y| = \lambda^n |x-y| \leq |x-y|$ for all $x, y \in K$ and $n \in \mathbf{N}$.

Thus T is an asymptotically nonexpansive mapping with constant sequence $\{k_n\} = \{1\}$ for all $n \geq 1$ and uniformly L-Lipschtzian mappings with $L = \sup_{n\geq 1}\{k_n\}$.

We know that the following iteration scheme for a mapping $T: K \to K$ are defined as follows:

Picard iteration scheme

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= T x_n,\ n \in \mathbf{N}. \end{aligned} \tag{1}$$

Mann iteration scheme

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T x_n,\ n \in \mathbf{N}, \end{aligned} \tag{2}$$

where $\{\alpha_n\}$ is a real sequence in (0,1).

In 2007, Agarwal *et al.* [1] introduced the following iteration scheme:

Modified S-iteration scheme

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n,\ n \in \mathbf{N}, \end{aligned} \tag{3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. They showed that this process converge at a rate same as that of Picard iteration and faster than Mann for contractions and also they established some weak convergence theorems using suitable conditions in the framework of uniformly convex Banach space.

In 2012, Guo *et al.* [4] studied the iteration scheme defined as follows:

Mixed type iteration scheme

$$
\begin{aligned}
x_1 &= x \in K, \\
x_{n+1} &= P((1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1} y_n), \\
y_n &= P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1} x_n),\ n \in \mathbf{N},
\end{aligned} \tag{4}
$$

where $S_1, S_2\colon K \to K$ are two asymptotically nonexpansive self mappings and $T_1, T_2\colon K \to E$ are two asymptotically nonexpansive non-self mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0,1)$, and proved some strong and weak convergence theorems for mixed type asymptotically nonexpansive mappings.

Recently, Wei and Guo [5] studied the iteration scheme defined as follows:

Mixed type iteration scheme with errors

Let E be a real Banach space, K be a nonempty closed convex subset of E and $P\colon E \to K$ is a nonexpansive retraction of E onto K. Let $S_1, S_2\colon K \to K$ be two asymptotically nonexpansive self mappings and $T_1, T_2\colon K \to E$ are two asymptotically nonexpansive non-self mappings. Then Wei and Guo [5] defined the new iteration scheme of mixed type with mean errors as follows:

$$
\begin{aligned}
x_1 &= x \in K, \\
x_{n+1} &= P(\alpha_n S_1^n x_n + \beta_n T_1(PT_1)^{n-1} y_n + \gamma_n u_n), \\
y_n &= P(\alpha'_n S_2^n x_n + \beta'_n T_2(PT_2)^{n-1} x_n + \gamma'_n u'_n),\ n \in \mathbf{N},
\end{aligned} \tag{5}
$$

where $\{u_n\}$, $\{u'_n\}$ are bounded sequences in E, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are real sequences in $[0,1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ for all $n \geq 1$, and proved some weak convergence theorems in the setting of real uniformly convex Banach spaces.

It is to be noted that (5) reduces to

- (4) when $\gamma_n = \gamma'_n = 0$ for all $n \in \mathbf{N}$.

Inspired and motivated by [4], [5] and some others, we proposed the following iteration scheme:

Hybrid mixed type iteration scheme with errors

Let E be a real Banach space, K be a nonempty closed convex subset of E and $P\colon E \to K$ is a nonexpansive retraction of E onto K. Let $S_1, S_2\colon K \to K$ be two nearly asymptotically nonexpansive self mappings and $T_1, T_2\colon K \to E$ are two asymptotically nonexpansive non-self mappings, then we defined the hybrid mixed type iteration scheme as follows:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= P((1-a_n-c_n)S_1^n x_n + a_n T_1(PT_1)^{n-1} y_n + c_n u_n), \\ y_n &= P((1-b_n-d_n)S_2^n x_n + b_n T_2(PT_2)^{n-1} x_n + d_n v_n),\ n \in \mathbf{N}, \end{aligned}$$

(6)

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ are four real sequences in $[0,1]$ satisfying $a_n + c_n \leq 1$, $b_n + d_n \leq 1$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K.

The aim of this paper is to study and establish some weak convergence theorems of iteration scheme (6) for mentioned scheme and mappings in the setting of uniformly convex Banach spaces. Our results extend and generalize several results from the current existing literature.

For the sake of convenience, we restate the following notion and results.

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is the function $\delta_E(\varepsilon)\colon (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\Big\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1,\ \|y\| = 1,\ \varepsilon = \|x-y\|\Big\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

Definition 1.3. Let $\mathscr{S} = \{x \in E : \|x\| = 1\}$ and let E^* be the dual of E, that is, the space of all continuous linear functionals f on E. The space E has:

(d_1) Gâteaux differentiable norm if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in $\mathscr{S}$.

(d_2) Fréchet differentiable norm [6] if for each x in $\mathscr{S}$, the above limit exists and is attained uniformly for y in $\mathscr{S}$ and in this case, it is also well-known that

$$\begin{aligned}\langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 &\leq \frac{1}{2}\|x+h\|^2 \\ &\leq \langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|x\|) \qquad (\mathbf{FDN})\end{aligned}$$

for all x, $h \in E$, where J is the Fréchet derivative of the functional $\frac{1}{2}\|.\|^2$ at $x \in E$, $\langle .\, .\rangle$ is the pairing between E and E^*, and b is an increasing function defined on $[0,\infty)$ such that $\lim_{t\to 0}\frac{b(t)}{t} = 0$.

(d_3) Opial condition [7] if for any sequence $\{x_n\}$ in E, x_n converges to x weakly it follows that $\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|$ for all $y \in E$ with $y \neq x$. Examples of Banach spaces satisfying Opial condition are Hilbert spaces and all spaces $l^p(1 < p < \infty)$. On the other hand, $L^p[0,2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial condition.

Definition 1.4. A mapping $T\colon K \to K$ is said to be demiclosed at zero, if for any sequence $\{x_n\}$ in K, the condition x_n converges weakly to $x \in K$ and Tx_n converges strongly to 0 imply $Tx = 0$.

Definition 1.5. A Banach space E has the Kadec-Klee property [8] if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ it follows that $\|x_n - x\| \to 0$.

Let δ be the modulus of uniform convexity. Recall that E is a uniformly convex Banach space then if (see [9])

$$\|tx + (1-t)y\| \leq 1 - 2t(1-t)\delta(\|x-y\|) \qquad (\mathbf{UCBS})$$

for all $t \in [0,1]$ and for all $x, y \in E$ such that $\|x\| \leq 1$, $\|y\| \leq 1$.

Next we state the following useful lemmas to prove our main results.

Lemma 1.1. [10] *Let* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ *and* $\{r_n\}_{n=1}^{\infty}$ *be sequences of nonnegative numbers satisfying the inequality* $\alpha_{n+1} \leq (1+\beta_n)\alpha_n + r_n$, $\forall n \geq 1$. *If* $\sum_{n=1}^{\infty}\beta_n < \infty$ *and* $\sum_{n=1}^{\infty} r_n < \infty$, *then*

(i) $\lim_{n\to\infty}\alpha_n$ *exists;*

(ii) In particular, if $\{\alpha_n\}_{n=1}^{\infty}$ *has a subsequence which converges strongly to zero, then* $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 1.2. [11] *Let E be a uniformly convex Banach space and* $0 < \alpha \leq t_n \leq \beta < 1$ *for all* $n \in \mathbb{N}$. *Suppose further that* $\{x_n\}$ *and* $\{y_n\}$ *are sequences of E such that* $\limsup_{n\to\infty}\|x_n\| \leq a$,

$\limsup_{n\to\infty}\|y_n\|\le a$ *and* $\lim_{n\to\infty}\|t_nx_n+(1-t_n)y_n\|=a$ *hold for some* $a\ge 0$. *Then* $\lim_{n\to\infty}\|x_n-y_n\|=0$.

Lemma 1.3. [8] *Let E be a real reflexive Banach space with its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and $p,q\in w_w(x_n)$ (where $w_w(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$). Suppose* $\lim_{n\to\infty}\|tx_n+(1-t)p-q\|$ *exists for all* $t\in[0,1]$. *Then $p=q$.*

Lemma 1.4. [8] *Let K be a nonempty convex subset of a uniformly convex Banach space E. Then there exists a strictly increasing continuous convex function $\phi\colon[0,\infty)\to[0,\infty)$ with $\phi(0)=0$ such that for each Lipschitzian mapping $T\colon K\to K$ with the Lipschitz constant L,*

$$\|tTx+(1-t)Ty-T(tx+(1-t)y)\|\le L\phi^{-1}\left(\|x-y\|-\frac{1}{L}\|Tx-Ty\|\right)$$

for all $x,y\in K$ and all $t\in[0,1]$.

2. Weak convergence theorems

In this section, we prove some convergence theorems of iteration scheme (6) for two nearly asymptotically nonexpansive self mappings and two asymptotically nonexpansive non-self mappings in real uniformly convex Banach spaces. First, we shall need the following lemmas.

Lemma 2.1. *Let E be a real Banach space, K be a nonempty closed convex subset of E. Let $S_1,S_2\colon K\to K$ be two nearly asymptotically nonexpansive self mappings with sequences $\{e'_n,k_n\}$, $\{e''_n,k_n\}$ such that $\sum_{n=1}^{\infty}e_n<\infty$, $\sum_{n=1}^{\infty}(k_n-1)<\infty$ and $T_1,T_2\colon K\to E$ are two asymptotically nonexpansive non-self mappings with a sequence $\{l_n\}\in[1,\infty)$ such that $\sum_{n=1}^{\infty}(l_n-1)<\infty$. Suppose that $F=F(S_1)\bigcap F(S_2)\bigcap F(T_1)\bigcap F(T_2)\neq\emptyset$. Let $M=\sup_n a_n$ and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ are four real sequences in $[0,1]$ which satisfy the following conditions:*

(i) $\sum_{n=1}^{\infty}c_n<\infty$, $\sum_{n=1}^{\infty}d_n<\infty$;

(ii) $M\rho<1$.

Let $\{x_n\}$ be the sequence defined by (6), then $\lim_{n\to\infty}\|x_n-q\|$ *and* $\lim_{n\to\infty}d(x_n,F)$ *both exist for each $q\in F$.*

Proof. Let $q \in F$. Let $e_n = \max\{e'_n . e''_n\}$ and $h_n = \max\{k_n, l_n\}$ with $\sum_{n=1}^{\infty} e_n < \infty$, $\sum_{n=1}^{\infty}(h_n - 1) < \infty$ and $\rho = \sup_n h_n$. From (6), we have

$$\begin{aligned}
\|y_n - q\| &\leq \|(1-b_n-d_n)S_2^n x_n + b_n T_2(PT_2)^{n-1}x_n + d_n v_n - q\| \\
&= \|(1-b_n-d_n)(S_2^n x_n - q) + b_n(T_2(PT_2)^{n-1}x_n - q) + d_n(v_n - q)\| \\
&\leq (1-b_n-d_n)\|S_2^n x_n - q\| + b_n\|T_2(PT_2)^{n-1}x_n - q\| + d_n\|v_n - q\| \\
&\leq (1-b_n-d_n)[k_n(\|x_n - q\| + e''_n)] + b_n l_n\|x_n - q\| + d_n\|v_n - q\| \\
&\leq (1-b_n-d_n)[h_n(\|x_n - q\| + e_n)] + b_n h_n\|x_n - q\| + d_n\|v_n - q\| \\
&\leq (1-b_n)[h_n(\|x_n - q\| + e_n)] + b_n h_n\|x_n - q\| + d_n\|v_n - q\| \\
&= [(1-b_n) + b_n]h_n\|x_n - q\| + d_n\|v_n - q\| + (1-b_n)e_n \\
&\leq h_n\|x_n - q\| + d_n\|v_n - q\| + e_n. \qquad (7)
\end{aligned}$$

Again using (7) and (8), we have

$$\begin{aligned}
\|x_{n+1} - q\| &\leq \|(1-a_n-c_n)S_1^n x_n + a_n T_1(PT_1)^{n-1}y_n + c_n u_n - q\| \\
&= \|(1-a_n-c_n)(S_1^n x_n - q) + a_n(T_1(PT_1)^{n-1}y_n - q) + c_n(u_n - q)\| \\
&\leq (1-a_n-c_n)\|S_1^n x_n - q\| + a_n\|T_1(PT_1)^{n-1}y_n - q\| + c_n\|u_n - q\| \\
&\leq (1-a_n-c_n)[k_n(\|x_n - q\| + e'_n)] + a_n l_n\|y_n - q\| + c_n\|u_n - q\| \\
&\leq (1-a_n-c_n)[h_n(\|x_n - q\| + e_n)] + a_n h_n\|y_n - q\| + c_n\|u_n - q\| \\
&\leq (1-a_n)[h_n(\|x_n - q\| + e_n)] + a_n h_n\|y_n - q\| + c_n\|u_n - q\| \\
&\leq (1-a_n)h_n\|x_n - q\| + a_n h_n\|y_n - q\| + e_n + c_n\|u_n - q\|. \qquad (8)
\end{aligned}$$

Using equation (7) in (8), we obtain

$$\begin{aligned}
\|x_{n+1} - q\| &\leq (1-a_n)h_n\|x_n - q\| + a_n h_n[h_n\|x_n - q\| \\
&\quad + d_n\|v_n - q\| + e_n] + c_n\|u_n - q\| + e_n \\
&\leq (1-a_n)h_n^2\|x_n - q\| + a_n h_n^2\|x_n - q\| + a_n h_n d_n\|v_n - q\| + c_n\|u_n - q\| + a_n h_n e_n + e_n \\
&= h_n^2\|x_n - q\| + a_n h_n d_n\|v_n - q\| + c_n\|u_n - q\| + (a_n h_n + 1)e_n \\
&= [1 + (h_n^2 - 1)]\|x_n - q\| + a_n h_n d_n\|v_n - q\| + c_n\|u_n - q\| + (a_n h_n + 1)e_n \\
&= [1 + \mu_n]\|x_n - q\| + \nu_n, \qquad (9)
\end{aligned}$$

where $\mu_n = (h_n^2 - 1)$ and $\nu_n = a_n h_n d_n \|v_n - q\| + c_n \|u_n - q\| + (a_n h_n + 1)e_n$. Hence, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \mu_n &= \sum_{n=1}^{\infty} (h_n^2 - 1) = \sum_{n=1}^{\infty} (h_n + 1)(h_n - 1) \\ &\leq (\rho + 1) \sum_{n=1}^{\infty} (h_n - 1) < \infty,\end{aligned}$$

and boundedness of the sequences $\{\|u_n - q\|\}$, $\{\|v_n - q\|\}$ with condition (i) of the lemma

$$\begin{aligned}\sum_{n=1}^{\infty} \nu_n &= \sum_{n=1}^{\infty} [a_n h_n d_n \|v_n - q\| + c_n \|u_n - q\| + (a_n h_n + 1)e_n] \\ &= \sum_{n=1}^{\infty} a_n h_n d_n \|v_n - q\| + \sum_{n=1}^{\infty} c_n \|u_n - q\| + \sum_{n=1}^{\infty} (a_n h_n + 1)e_n \\ &\leq M\rho \sum_{n=1}^{\infty} d_n \|v_n - q\| + \sum_{n=1}^{\infty} c_n \|u_n - q\| \\ &\quad (M\rho + 1) \sum_{n=1}^{\infty} e_n < \infty.\end{aligned}$$

Now taking $s_n = \|x_n - q\|$ in (9), we obtain $s_{n+1} \leq (1 + \mu_n)s_n + \nu_n$. Hence by Lemma 1.1 (i), we have $\lim_{n\to\infty} \|x_n - q\|$ exists.

Now, taking the infimum over all $q \in F$ in (9), we have

$$d(x_{n+1}, F) \leq [1 + \mu_n] d(x_n, F) + \nu_n \tag{10}$$

for all $n \in \mathbf{N}$. It follows from $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$ and Lemma 1.1 (i) that $\lim_{n\to\infty} d(x_n, F)$ exists. This completes the proof.

Lemma 2.2. *Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E. Let $S_1, S_2 \colon K \to K$ be two nearly asymptotically nonexpansive self mappings with sequences $\{e'_n, k_n\}$, $\{e''_n, k_n\}$ such that $\sum_{n=1}^{\infty} e_n < \infty$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $T_1, T_2 \colon K \to E$ are two asymptotically nonexpansive non-self mappings with a sequence $\{l_n\} \in [1, \infty)$ such that $\sum_{n=1}^{\infty} (l_n - 1) < \infty$. Suppose that $F = F(S_1) \bigcap F(S_2) \bigcap F(T_1) \bigcap F(T_2) \neq \emptyset$. Let $M = \sup_n a_n$ and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ are four real sequences in $[0, 1]$ which satisfy the following conditions:*

(i) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$;

(iii) $M\rho < 1$, where ρ is taken as in Lemma 2.1;

(iii) $\|x - T_1(PT_1)^{n-1}y\| \leq \|S_1^n x - T_1(PT_1)^{n-1}y\|$ and $\|x - T_i(PT_i)^{n-1}x\|$ $\leq \|S_i^n x - T_i(PT_i)^{n-1}x\|$ for all $x, y \in K$ and for $i = 1, 2$.

Let $\{x_n\}$ be the sequence defined by (6). Then $\lim_{n\to\infty}\|x_n - S_i x_n\| = 0$ *and* $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ *for* $i = 1, 2$.

Proof. By Lemma 2.1, $\lim_{n\to\infty}\|x_n - q\|$ exists for all $q \in F$ and therefore $\{x_n\}$ is bounded. Let $\lim_{n\to\infty}\|x_n - q\| = z$. Then $z > 0$ otherwise there is nothing to prove. Now (7) implies that

$$\limsup_{n\to\infty}\|y_n - q\| \;\leq\; z. \tag{11}$$

Also, we have

$$\|S_2^n x_n - q\| \leq k_n(\|x_n - q\| + e''_n) \leq h_n(\|x_n - q\| + e_n),\ \forall n \geq 1,$$

$$\|T_2(PT_2)^{n-1}x_n - q\| \leq l_n\|x_n - q\| \leq h_n\|x_n - q\|,\ \forall n \geq 1,$$

and

$$\|S_1^n x_n - q\| \leq k_n(\|x_n - q\| + e'_n) \leq h_n(\|x_n - q\| + e_n),\ \forall n \geq 1.$$

Hence, we have

$$\limsup_{n\to\infty}\|S_2^n x_n - q\| \;\leq\; z, \tag{12}$$

$$\limsup_{n\to\infty}\|T_2(PT_2)^{n-1}x_n - q\| \;\leq\; z, \tag{13}$$

and

$$\limsup_{n\to\infty}\|S_1^n x_n - q\| \;\leq\; z. \tag{14}$$

Note that $\|T_1(PT_1)^{n-1}y_n - q\| \leq l_n\|y_n - q\| \leq h_n\|y_n - q\|$ By virtue of (11), we find that

$$\limsup_{n\to\infty}\|T_1(PT_1)^{n-1}y_n - q\| \;\leq\; z. \tag{15}$$

Also, it follows from

$$\begin{aligned}
z &= \lim_{n\to\infty}\|x_{n+1} - q\| \\
&= \lim_{n\to\infty}\|(1 - a_n - c_n)S_1^n x_n + a_n T_1(PT_1)^{n-1}y_n + c_n u_n - q\| \\
&= \lim_{n\to\infty}\|(1 - a_n)[(S_1^n x_n - q) + c_n(u_n - S_1^n x_n)] \\
&\quad + a_n[(T_1(PT_1)^{n-1}y_n - q) + c_n(u_n - S_1^n x_n)]\|
\end{aligned}$$

and Lemma 1.2 that

$$\lim_{n\to\infty}\|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| = 0. \tag{16}$$

By condition (iii), it follows that $\|x_n - T_1(PT_1)^{n-1}y_n\| \leq \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\|$. From (16), we have

$$\lim_{n\to\infty} \|x_n - T_1(PT_1)^{n-1}y_n\| = 0. \tag{17}$$

Since

$$\begin{aligned}
\|x_n - q\| &\leq \|x_n - T_1(PT_1)^{n-1}y_n\| + \|T_1(PT_1)^{n-1}y_n - q\| \\
&\leq \|x_n - T_1(PT_1)^{n-1}y_n\| + l_n\|y_n - q\| \\
&\leq \|x_n - T_1(PT_1)^{n-1}y_n\| + h_n\|y_n - q\|,
\end{aligned}$$

and taking limitinf on both sides in the above inequality, we have $\liminf_{n\to\infty} \|y_n - q\| \geq z$. By (17), we have

$$\lim_{n\to\infty} \|y_n - q\| = z. \tag{18}$$

Now, note that

$$\begin{aligned}
z &= \lim_{n\to\infty} \|y_n - q\| \\
&= \lim_{n\to\infty} \|(1-b_n-d_n)S_2^n x_n + b_n T_2(PT_2)^{n-1}x_n + d_n u_n - q\| \\
&= \lim_{n\to\infty} \|(1-b_n)[(S_2^n x_n - q) + d_n(v_n - S_2^n x_n)] + b_n[(T_2(PT_2)^{n-1}x_n - q) + d_n(v_n - S_2^n x_n)]\|.
\end{aligned}$$

It follows from Lemma 1.2 that

$$\lim_{n\to\infty} \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{19}$$

By condition (iii), we have $\|x_n - T_2(PT_2)^{n-1}x_n\| \leq \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\|$. In view of (19), we have

$$\lim_{n\to\infty} \|x_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{20}$$

Since $S_2^n x_n = P(S_2^n x_n)$ and $P\colon E \to K$ is a nonexpansive retraction of E onto K, we have

$$\begin{aligned}
\|y_n - S_2^n x_n\| &= \|(1-b_n-d_n)S_2^n x_n + b_n T_2(PT_2)^{n-1}x_n + d_n v_n - S_2^n x_n\| \\
&= \|b_n(S_2^n x_n - T_2(PT_2)^{n-1}x_n) + d_n(v_n - S_2^n x_n)\| \\
&\leq b_n\|S_2^n x_n - T_2(PT_2)^{n-1}x_n\| + d_n\|v_n - S_2^n x_n\|
\end{aligned}$$

and so

$$\lim_{n\to\infty} \|y_n - S_2^n x_n\| = 0. \tag{21}$$

Again, we have

$$\|y_n - x_n\| \leq \|y_n - S_2^n x_n\| + \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\| + \|T_2(PT_2)^{n-1}x_n - x_n\|.$$

Thus, it follows from (19), (20) and (21) that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{22}$$

Note that $\|x_n - T_1(PT_1)^{n-1}y_n\| \leq \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\|$. By condition (iii),

$$\begin{aligned}
\|S_1^n x_n - T_1(PT_1)^{n-1}x_n\| &\leq \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + \|T_1(PT_1)^{n-1}y_n - T_1(PT_1)^{n-1}x_n\| \\
&\leq \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + l_n\|y_n - x_n\| \\
&\leq \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + h_n\|y_n - x_n\|,
\end{aligned}$$

and using (16), (22) and $h_n \to 1$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|S_1^n x_n - T_1(PT_1)^{n-1}x_n\| = 0. \tag{23}$$

It follows from condition (iii) that

$$\lim_{n\to\infty} \|x_n - T_1(PT_1)^{n-1}x_n\| = 0. \tag{24}$$

It follows from

$$\begin{aligned}
\|x_{n+1} - S_1^n x_n\| &= \|P((1-a_n-c_n)S_1^n x_n + a_n T_1(PT_1)^{n-1}y_n + c_n u_n) - P(S_1^n x_n)\| \\
&\leq \|(1-a_n-c_n)S_1^n x_n + a_n T_1(PT_1)^{n-1}y_n + c_n u_n - S_1^n x_n\| \\
&\leq a_n\|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + c_n\|u_n - S_1^n x_n\| \\
&\leq M\|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + c_n\|u_n - S_1^n x_n\|,
\end{aligned}$$

(16) and $c_n \to 0$ as $n \to \infty$ that

$$\lim_{n\to\infty} \|x_{n+1} - S_1^n x_n\| = 0. \tag{25}$$

In addition, we have

$$\|x_{n+1} - T_1(PT_1)^{n-1}y_n\| \leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\|.$$

Using (16) and (25), we have

$$\lim_{n\to\infty} \|x_{n+1} - T_1(PT_1)^{n-1}y_n\| = 0. \tag{26}$$

Now, using (23), (24) and the inequality

$$\|S_1^n x_n - x_n\| \leq \|S_1^n x_n - T_1(PT_1)^{n-1}x_n\| + \|T_1(PT_1)^{n-1}x_n - x_n\|,$$

we have $\lim_{n\to\infty} \|S_1^n x_n - x_n\| = 0$. It follows from (20) and the inequality

$$\|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| \leq \|S_1^n x_n - x_n\| + \|x_n - T_2(PT_2)^{n-1}x_n\|$$

that

$$\lim_{n\to\infty} \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{27}$$

Since

$$\begin{aligned}\|x_{n+1} - T_2(PT_2)^{n-1}y_n\| &\leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| + l_n\|x_n - y_n\| \\ &\leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| + h_n\|x_n - y_n\|,\end{aligned}$$

from (22), (25), (27) and $h_n \to 1$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \|x_{n+1} - T_2(PT_2)^{n-1}y_n\| = 0. \tag{28}$$

Since T_i for $i = 1,2$ is uniformly continuous, P is nonexpansive retraction, it follows from (28) that

$$\begin{aligned}\|T_i(PT_i)^{n-1}y_{n-1} - T_i x_n\| &= \|T_i[(PT_i)(PT)^{n-2})y_{n-1}] - T_i(Px_n)\| \\ &\to 0 \text{ as } n \to \infty, \end{aligned} \tag{29}$$

for $i = 1,2$. Moreover, we have

$$\|x_{n+1} - y_n\| \leq \|x_{n+1} - T_1(PT_1)^{n-1}y_n\| + \|T_1(PT_1)^{n-1}y_n - x_n\| + \|x_n - y_n\|.$$

Using (17), (22) and (26), we have

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{30}$$

In addition, we have

$$\begin{aligned}\|x_n - T_1x_n\| &\leq \|x_n - T_1(PT_1)^{n-1}x_n\| + \|T_1(PT_1)^{n-1}x_n - T_1(PT_1)^{n-1}y_{n-1}\| \\ &\quad + \|T_1(PT_1)^{n-1}y_{n-1} - T_1x_n\| \\ &\leq \|x_n - T_1(PT_1)^{n-1}x_n\| + l_n\|x_n - y_{n-1}\| + \|T_1(PT_1)^{n-1}y_{n-1} - T_1x_n\| \\ &\leq \|x_n - T_1(PT_1)^{n-1}x_n\| + h_n\|x_n - y_{n-1}\| + \|T_1(PT_1)^{n-1}y_{n-1} - T_1x_n\|.\end{aligned}$$

Thus, it follows from (24), (29), (30) and $h_n \to 1$ as $n \to \infty$, that

$$\lim_{n\to\infty} \|x_n - T_1x_n\| = 0. \tag{31}$$

Similarly, we can prove that

$$\lim_{n\to\infty} \|x_n - T_2x_n\| = 0. \tag{32}$$

Finally, we have

$$\begin{aligned}\|x_n - S_1x_n\| &\leq \|x_n - T_1(PT_1)^{n-1}x_n\| + \|S_1x_n - T_1(PT_1)^{n-1}x_n\| \\ &\leq \|x_n - T_1(PT_1)^{n-1}x_n\| + \|S_1^nx_n - T_1(PT_1)^{n-1}x_n\| \\ &\quad (\text{by cond. (iii)}).\end{aligned}$$

Thus, it follows from (23) and (24) that

$$\lim_{n\to\infty} \|x_n - S_1x_n\| = 0. \tag{33}$$

Similarly, we can prove that

$$\lim_{n\to\infty} \|x_n - S_2x_n\| = 0. \tag{34}$$

This completes the proof.

Lemma 2.3. *Under the assumptions of Lemma 2.1, for all $q_1, q_2 \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$, the limit $\lim_{n\to\infty} \|tx_n + (1-t)q_1 - q_2\|$ exists for all $t \in [0,1]$, where $\{x_n\}$ is the sequence defined by (6).*

Proof. By Lemma 2.1, $\lim_{n\to\infty} \|x_n - z\|$ exists for all $z \in F$ and therefore $\{x_n\}$ is bounded. Letting $a_n(t) = \|tx_n + (1-t)q_1 - q_2\|$ for all $t \in [0,1]$. Then $\lim_{n\to\infty} a_n(0) = \|q_1 - q_2\|$ and $\lim_{n\to\infty} a_n(1) = \|x_n - q_2\|$ exists by Lemma 2.1. It, therefore, remains to prove the Lemma 2.3 for $t \in (0,1)$. For all $x \in K$, we define the mapping $\mathscr{R}_n \colon K \to K$ by $\mathscr{R}_n(x) = P((1-a_n-c_n)S_1^nx+$

$a_nT_1(PT_1)^{n-1}\mathscr{A}_n(x)+c_nu_n)$ where $\mathscr{A}_n(x)=P((1-b_n-d_n)S_2^nx+b_nT_2(PT_2)^{n-1}x+d_nv_n)$. Then it follows that $x_{n+1}=\mathscr{R}_nx_n$, $\mathscr{R}_np=p$ for all $p\in F$. Now from (7) and (9) of Lemma 2.1, we see that $\|\mathscr{A}_n(x)-\mathscr{A}_n(y)\|\leq h_n(\|x-y\|+e_n)$ and

$$\begin{aligned}\|\mathscr{R}_n(x)-\mathscr{R}_n(y)\| &\leq [1+\mu_n]\|x-y\|+\phi_n\\ &= \mathscr{I}_n\|x-y\|+\phi_n,\end{aligned}\tag{35}$$

where $\mu_n=(h_n^2-1)$ and $\phi_n=(1+a_nh_n)h_ne_n$ with $\sum_{n=1}^{\infty}\mu_n<\infty$, $\sum_{n=1}^{\infty}\phi_n<\infty$, $\mathscr{I}_n=1+\mu_n$ and $\mathscr{I}_n\to 1$ as $n\to\infty$. Setting

$$S_{n,m}=\mathscr{R}_{n+m-1}\mathscr{R}_{n+m-2}\ldots\mathscr{R}_n,\ m\geq 1.\tag{36}$$

From (35) and (36), we have

$$\begin{aligned}\|S_{n,m}(x)-S_{n,m}(y)\| &\leq \mathscr{I}_{n+m-1}\|\mathscr{R}_{n+m-2}\ldots\mathscr{R}_n(x)-\mathscr{R}_{n+m-2}\ldots\mathscr{R}_n(y)\|+\phi_{n+m-1}\\ &\leq \mathscr{I}_{n+m-1}\mathscr{I}_{n+m-2}\|\mathscr{R}_{n+m-3}\ldots\mathscr{R}_n(x)-\mathscr{R}_{n+m-3}\ldots\mathscr{R}_n(y)\|\\ &\quad +\phi_{n+m-1}+\phi_{n+m-2}\\ &\ \vdots\\ &\leq \Big(\prod_{i=n}^{n+m-1}\mathscr{I}_i\Big)\|x-y\|+\sum_{i=n}^{n+m-1}\phi_i\\ &= \mathscr{G}_{n,m}\|x-y\|+f_{n,m}\end{aligned}\tag{37}$$

for all $x,y\in K$, where $\mathscr{G}_{n,m}=\prod_{i=n}^{n+m-1}\mathscr{I}_i$, $f_{n,m}=\sum_{i=n}^{n+m-1}\phi_i$, $S_{n,m}x_n=x_{n+m}$ and $S_{n,m}p=p$ for all $p\in F$.

For the sake of simplicity, set

$$
\begin{aligned}
f_{n,m} &= \sum_{i=n}^{n+m-1} \phi_i,\ f_n = \sum_{i=n}^{\infty} \phi_i,\\
\mathscr{G}_{n,m} &= \Big(\prod_{i=n}^{n+m-1} \mathscr{I}_i\Big),\ \mathscr{G}_n = \Big(\prod_{i=n}^{\infty} \mathscr{I}_i\Big),\\
t_n &= t x_n + (1-t) q_1,\\
\delta_{n,m} &= t\mathscr{G}_{n,m}\|x_n - q_1\| + f_{n,m},\\
\rho_{n,m} &= (1-t)\mathscr{G}_{n,m}\|x_n - q_1\| + f_{n,m},\\
v_{n,m} &= [t q_1 + (1-2t) S_{n,m} t_n - (1-t) S_{n,m} x_n],\\
\mathscr{B}_{n,m} &= [S_{n,m} t_n - t S_{n,m} x_n - (1-t) q_1)]\|x_n - q_1\|,\\
\mathscr{D}_{n,m} &= [q_1 + S_{n,m} x_n - 2 S_{n,m} t_n] f_{n,m},\\
\mathscr{W}_{n,m} &= (q_1 - S_{n,m} t_n)/\delta_{n,m},\\
\mathscr{Z}_{n,m} &= (S_{n,m} t_n - S_{n,m} x_n)/\rho_{n,m},\\
\lambda_{n,m} &= \delta_{n,m}\rho_{n,m}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|\mathscr{W}_{n,m}\| &= \Big\|\frac{S_{n,m} t_n - q_1}{t\mathscr{G}_{n,m}\|x_n - q_1\| + f_{n,m}}\Big\|\\
&\le \frac{t\mathscr{G}_{n,m}\|x_n - q_1\| + f_{n,m}}{t\mathscr{G}_{n,m}\|x_n - q_1\| + f_{n,m}} = 1.
\end{aligned}
$$

Similarly, we have $\|\mathscr{Z}_{n,m}\| \le 1$. Notice that

$$
\begin{aligned}
\delta_{n,m} + \rho_{n,m} &= \mathscr{G}_{n,m}[t\|x_n - q_1\| + (1-t)\|x_n - q_1\|] + 2 f_{n,m}\\
&= \mathscr{G}_{n,m}\|x_n - q_1\| + 2 f_{n,m}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\|\mathscr{W}_{n,m}-\mathscr{Z}_{n,m}\| &= \left\|\frac{q_1-S_{n,m}t_n}{\delta_{n,m}}-\frac{S_{n,m}t_n-S_{n,m}x_n}{\rho_{n,m}}\right\| \\
&= \left\|\frac{q_1\rho_{n,m}-\rho_{n,m}S_{n,m}t_n-\delta_{n,m}S_{n,m}t_n+\delta_{n,m}S_{n,m}x_n}{\lambda_{n,m}}\right\| \\
&= \left\|\frac{q_1\rho_{n,m}-[\mathscr{G}_{n,m}\|x_n-q_1\|+2f_{n,m}]S_{n,m}t_n+\delta_{n,m}S_{n,m}x_n}{\lambda_{n,m}}\right\| \\
&= \left\|\frac{\mathscr{B}_{n,m}-\mathscr{D}_{n,m}}{\lambda_{n,m}}\right\|
\end{aligned}
$$

because

$$
\begin{aligned}
\|\mathscr{B}_{n,m}-\mathscr{D}_{n,m}\| &= \Big\|\|x_n-q_1\|S_{n,m}t_n-\|x_n-q_1\|tS_{n,m}x_n-(1-t)q_1\|x_n-q_1\| \\
&\qquad -q_1f_{n,m}-f_{n,m}S_{n,m}x_n+2f_{n,m}S_{n,m}t_n\Big\| \\
&= \Big\|-[(1-t)\|x_n-q_1\|+f_{n,m}]q_1+[\|x_n-q_1\|+2f_{n,m}]S_{n,m}t_n \\
&\qquad -[t\|x_n-q_1\|+f_{n,m}]S_{n,m}x_n\Big\|
\end{aligned}
$$

and

$$
\begin{aligned}
\|t\mathscr{W}_{n,m}+(1-t)\mathscr{Z}_{n,m}\| &= \left\|\frac{t(q_1-S_{n,m}t_n)}{\delta_{n,m}}-\frac{(1-t)(S_{n,m}t_n-S_{n,m}x_n)}{\rho_{n,m}}\right\| \\
&= \left\|\frac{t\rho_{n,m}(q_1-S_{n,m}t_n)+(1-t)\delta_{n,m}(S_{n,m}t_n-S_{n,m}x_n)}{\lambda_{n,m}}\right\| \\
&= \frac{1}{\lambda_{n,m}}\Big\|[(1-t)\mathscr{G}_{n,m}\|x_n-q_1\|+f_{n,m}]t(q_1-S_{n,m}t_n) \\
&\qquad +[t\mathscr{G}_{n,m}\|x_n-q_1\|+f_{n,m}](1-t)(S_{n,m}t_n-S_{n,m}x_n)\Big\| \\
&= \frac{1}{\lambda_{n,m}}\Big\|t(1-t)\mathscr{G}_{n,m}\|x_n-q_1\|+tq_1f_{n,m} \\
&\qquad -t(1-t)\mathscr{G}_{n,m}S_{n,m}t_n\|x_n-q_1\|-tS_{n,m}t_nf_{n,m} \\
&\qquad +t(1-t)\mathscr{G}_{n,m}\|x_n-q_1\|S_{n,m}t_n+(1-t)f_{n,m}S_{n,m}t_n \\
&\qquad -t(1-t)\mathscr{G}_{n,m}\|x_n-q_1\|S_{n,m}x_n-(1-t)f_{n,m}S_{n,m}x_n\Big\| \\
&= \frac{1}{\lambda_{n,m}}\Big\|t(1-t)\mathscr{G}_{n,m}\|x_n-q_1\|(q_1-S_{n,m}x_n) \\
&\qquad +[tq_1+(1-2t)S_{n,m}t_n-(1-t)S_{n,m}x_n]f_{n,m}\Big\| \\
&= \frac{1}{\lambda_{n,m}}\Big\|t(1-t)\mathscr{G}_{n,m}\|x_n-q_1\|(q_1-x_{n+m})+v_{n,m}f_{n,m}\Big\|.
\end{aligned}
$$

From inequality (**UCBS**), we get

$$\begin{aligned} 2t(1-t)\lambda_{n,m}\delta\Big(\frac{\|\mathscr{B}_{n,m}-\mathscr{D}_{n,m}\|}{\lambda_{n,m}}\Big) &= \lambda_{n,m}-\|t(1-t)\|x_n-q_1\|(q_1-x_{n+m}) \\ &\qquad +v_{n,m}f_{n,m}\| \\ &\leq \lambda_{n,m}-t(1-t)\|x_n-q_1\|\|x_{n+m}-q_1\| \\ &\quad +\|v_{n,m}\|f_{n,m}. \end{aligned}$$

But

$$\begin{aligned} \lambda_{n,m} &= \Big(t\mathscr{G}_{n,m}\|x_n-q_1\|+f_{n,m}\Big)\Big((1-t)\mathscr{G}_{n,m}\|x_n-q_1\|+f_{n,m}\Big) \\ &= \mathscr{G}_{n,m}^2 t(1-t)\|x_n-q_1\|^2+t\mathscr{G}_{n,m}f_{n,m}\|x_n-q_1\| \\ &\quad +(1-t)\mathscr{G}_{n,m}f_{n,m}\|x_n-q_1\|+f_{n,m}^2 \\ &= \mathscr{G}_{n,m}^2 t(1-t)\|x_n-q_1\|^2+\mathscr{G}_{n,m}f_{n,m}\|x_n-q_1\|+f_{n,m}^2 \\ &\leq \mathscr{G}_n^2 t(1-t)\|x_n-q_1\|^2+[\mathscr{G}_n\|x_n-q_1\|+f_n]f_n \\ &\leq \mathscr{G}_n^2 t(1-t)\|x_n-q_1\|^2+\mathscr{K}_1 f_n, \end{aligned}$$

where $\mathscr{K}_1=\sup_n\{\mathscr{G}_n\|x_n-q_1\|+f_n\}$. Therefore

$$\begin{aligned} 2\lambda_{n,m}\delta\Big(\frac{\|\mathscr{B}_{n,m}-\mathscr{D}_{n,m}\|}{\lambda_{n,m}}\Big) &\leq \mathscr{G}_n^2\|x_n-q_1\|^2+\frac{\mathscr{K}_1 f_n}{t(1-t)} \\ &\qquad -\|x_n-q_1\|\|q_1-x_{n+m}\|+\frac{\|v_{n,m}\|f_{n,m}}{t(1-t)}. \end{aligned}$$

Let $\lambda=\sup\{\lambda_n\mathscr{G}_n : n\in\mathbf{N}\}$. Since E is uniformly convex $\delta(s)/s$ is nondecreasing. Therefore

$$\begin{aligned} 2\lambda\,\delta\Big(\frac{\|\mathscr{B}_{n,m}-\mathscr{D}_{n,m}\|}{\lambda}\Big) &\leq \mathscr{G}_n^2\|x_n-q_1\|^2+\frac{\mathscr{K}_1 f_n}{t(1-t)} \\ &\qquad -\|x_n-q_1\|\|q_1-x_{n+m}\|+\frac{\|v_{n,m}\|f_{n,m}}{t(1-t)}. \end{aligned}$$

Moreover $\delta(0)=0$, $\lim_{n\to\infty}f_n=0$, $\lim_{n\to\infty}\mathscr{G}_n=1$ and δ is continuous, therefore

$$\lim_{m,n\to\infty}\|\mathscr{B}_{n,m}-\mathscr{D}_{n,m}\|=0.$$

By the triangle inequality, we have

$$\begin{aligned} \|\mathscr{B}_{n,m}\| &\leq \|\mathscr{B}_{n,m}-\mathscr{D}_{n,m}\|+\|\mathscr{D}_{n,m}\| \\ &= \|\mathscr{B}_{n,m}-\mathscr{D}_{n,m}\|+\mathscr{K}_2 f_{n,m} \end{aligned}$$

for some $\mathscr{K}_2 > 0$. This gives $\lim_{m,n\to\infty} \|\mathscr{B}_{n,m}\| = 0$. Since $\lim_{n\to\infty} \|x_n - q_1\| > 0$, we have $\lim_{m,n\to\infty} \|S_{n,m}t_n - tS_{n,m}x_n - (1-t)q_1\| = 0$. Finally, from

$$\begin{aligned} a_{n+m}(t) &= \|tx_{n+m} + (1-t)q_1 - q_2\| \\ &\leq \|S_{n,m}t_n - q_2\| + \|S_{n,m}t_n - tS_{n,m}x_n - (1-t)q_1\| \\ &\leq \mathscr{G}_{n,m}\|t_n - q_2\| + f_{n,m} + \|S_{n,m}t_n - tS_{n,m}x_n - (1-t)q_1\| \\ &\leq \mathscr{G}_n\|t_n - q_2\| + f_n + \|S_{n,m}t_n - tS_{n,m}x_n - (1-t)q_1\|, \end{aligned}$$

we get

$$\begin{aligned} \limsup_{m\to\infty} a_{n+m}(t) &\leq \liminf_{n\to\infty} \Big(\mathscr{G}_n\|t_n - q_2\| + f_n\Big) \\ &\quad + \limsup_{m\to\infty} \Big(\|S_{n,m}t_n - tS_{n,m}x_n - (1-t)q_1\|\Big) \\ &= \liminf_{n\to\infty} a_n(t). \end{aligned}$$

Thus $\limsup_{n\to\infty} a_n(t) \leq \liminf_{n\to\infty} a_n(t)$. It follows that

$$\lim_{n\to\infty} \|tx_n + (1-t)q_1 - q_2\|$$

exists for all $t \in [0,1]$. This completes the proof.

Lemma 2.4. *Under the assumptions of Lemma 2.1, if E has a Frěchet differentiable norm, then for all $q_1, q_2 \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$, the limit $\lim_{n\to\infty}\langle x_n, J(q_1 - q_2)\rangle$ exists, where $\{x_n\}$ is the sequence defined by (6), if $W_w(\{x_n\})$ denotes the set of all weak subsequential limits of $\{x_n\}$, then $\langle u_1 - u_2, J(q_1 - q_2)\rangle = 0$ for all $q_1, q_2 \in F$ and $u_1, u_2 \in W_w(\{x_n\})$.*

Proof. Suppose that $x = q_1 - q_2$ with $q_1 \neq q_2$ and $h = t(x_n - q_1)$ in inequality (**FDN**). Then, we get

$$\begin{aligned} & t\,\langle x_n - q_1, J(q_1 - q_2)\rangle + \frac{1}{2}\|q_1 - q_2\|^2 \\ \leq\ & \frac{1}{2}\|tx_n + (1-t)q_1 - q_2\|^2 \\ \leq\ & t\,\langle x_n - q_1, J(q_1 - q_2)\rangle + \frac{1}{2}\|q_1 - q_2\|^2 \\ & + b(t\|x_n - q_1\|). \end{aligned}$$

Since $\sup_{n\geq 1}\|x_n-q_1\|\leq \mathscr{R}$ for some $\mathscr{R}>0$, we have

$$\begin{aligned} & t\limsup_{n\to\infty}\langle x_n-q_1, J(q_1-q_2)\rangle+\frac{1}{2}\|q_1-q_2\|^2 \\ \leq\ & \frac{1}{2}\lim_{n\to\infty}\|tx_n+(1-t)q_1-q_2\|^2 \\ \leq\ & t\liminf_{n\to\infty}\langle x_n-q_1, J(q_1-q_2)\rangle+\frac{1}{2}\|q_1-q_2\|^2 \\ & +b(t\mathscr{R}). \end{aligned}$$

That is,

$$\begin{aligned} & \limsup_{n\to\infty}\langle x_n-q_1, J(q_1-q_2)\rangle \\ \leq\ & \liminf_{n\to\infty}\langle x_n-q_1, J(q_1-q_2)\rangle+\frac{b(t\mathscr{R})}{t\mathscr{R}}\mathscr{R}. \end{aligned}$$

If $t\to 0$, then $\lim_{n\to\infty}\langle x_n-q_1, J(q_1-q_2)\rangle$ exists for all $q_1,q_2\in F$; in particular, we have $\langle u_1-u_2, J(q_1-q_2)\rangle=0$ for all $u_1,u_2\in W_w(\{x_n\})$. This completes the proof.

Theorem 2.1. *Under the assumptions of Lemma 2.2, if E has Frěchet differentiable norm, then the sequence $\{x_n\}$ defined by (6) converges weakly to a common fixed point of S_1, S_2, T_1 and T_2.*

Proof. By Lemma 2.4, $\langle u_1-u_2, J(q_1-q_2)\rangle=0$ for all $u_1,u_2\in W_w(\{x_n\})$. Therefore $\|q^*-p^*\|^2=\langle q^*-p^*, J(q^*-p^*)\rangle=0$ implies $q^*=p^*$. Consequently, $\{x_n\}$ converges weakly to a common fixed point in $F=F(S_1)\cap F(S_2)\cap F(T_1)\cap F(T_2)$. This completes the proof.

Theorem 2.2. *Under the assumptions of Lemma 2.2, if the dual space E^* of E has the Kadec-Klee (KK) property and the mappings $I-S_i$ and $I-T_i$ for $i=1,2$, where I denotes the identity mapping, are demiclosed at zero, then the sequence $\{x_n\}$ defined by (6) converges weakly to a common fixed point of S_1, S_2, T_1 and T_2.*

Proof. By Lemma 2.1, $\{x_n\}$ is bounded and since E is reflexive, there exists a subsequence $\{x_{n_r}\}$ of $\{x_n\}$ which converges weakly to some $u^*\in K$. By Lemma 2.2, we have

$$\lim_{r\to\infty}\|x_{n_r}-S_ix_{n_r}\|=0 \text{ and } \lim_{r\to\infty}\|x_{n_r}-T_ix_{n_r}\|=0$$

for $i=1,2$. Since by hypothesis the mappings $I-S_i$ and $I-T_i$ for $i=1,2$ are demiclosed at zero, therefore $S_iu^*=u^*$ and $T_iu^*=u^*$ for $i=1,2$, which means $u^*\in F=F(S_1)\cap F(S_2)\cap F(T_1)\cap F(T_2)$. Now, we show that $\{x_n\}$ converges weakly to u^*. Suppose $\{x_{n_s}\}$ is another subsequence of $\{x_n\}$ converges weakly to some $v^*\in K$. By the same method as above, we have

$v^* \in F$ and $u^*, v^* \in W_w(\{x_n\})$. By Lemma 2.3, the limit $\lim_{n\to\infty} \|tx_n + (1-t)u^* - v^*\|$ exists for all $t \in [0,1]$ and so $u^* = v^*$ by Lemma 1.3. Thus, the sequence $\{x_n\}$ converges weakly to $u^* \in F$. This completes the proof.

Theorem 2.3. *Under the assumptions of Lemma 2.2, if E satisfies Opial's condition and the mappings $I - S_i$ and $I - T_i$ for $i = 1,2$, where I denotes the identity mapping, are demiclosed at zero, then the sequence $\{x_n\}$ defined by (6) converges weakly to a common fixed point of S_1, S_2, T_1 and T_2.*

Proof. Let $g_* \in F$, from Lemma 2.1 the sequence $\{\|x_n - g_*\|\}$ is convergent and hence bounded. Since E is uniformly convex, every bounded subset of E is weakly compact. Thus there exists a subsequence $\{x_{n_r}\} \subset \{x_n\}$ such that $\{x_{n_r}\}$ converges weakly to $f^* \in K$. From Lemma 2.2, we have $\lim_{r\to\infty} \|x_{n_r} - S_i x_{n_r}\| = 0$ and $\lim_{r\to\infty} \|x_{n_r} - T_i x_{n_r}\| = 0$ for $i = 1,2$. Since the mappings $I - S_i$ and $I - T_i$ for $i = 1,2$ are demiclosed at zero, therefore $S_i f^* = f^*$ and $T_i f^* = f^*$ for $i = 1,2$, which means $f^* \in F$. Finally, let us prove that $\{x_n\}$ converges weakly to f^*. Suppose on contrary that there is a subsequence $\{x_{n_s}\} \subset \{x_n\}$ such that $\{x_{n_s}\}$ converges weakly to $h^* \in K$ and $f^* \neq h^*$. Then by the same method as given above, we can also prove that $h^* \in F$. From Lemma 2.1 the limits $\lim_{n\to\infty} \|x_n - f^*\|$ and $\lim_{n\to\infty} \|x_n - h^*\|$ exist. By virtue of the Opial condition of E, we obtain

$$
\begin{aligned}
\lim_{n\to\infty} \|x_n - f^*\| &= \lim_{n_r\to\infty} \|x_{n_r} - f^*\| \\
&< \lim_{n_r\to\infty} \|x_{n_r} - h^*\| \\
&= \lim_{n\to\infty} \|x_n - h^*\| \\
&= \lim_{n_s\to\infty} \|x_{n_s} - h^*\| \\
&< \lim_{n_s\to\infty} \|x_{n_s} - f^*\| \\
&= \lim_{n\to\infty} \|x_n - f^*\|,
\end{aligned}
$$

which derives a contradiction, so $f^* = h^*$. Thus $\{x_n\}$ converges weakly to a common fixed point of S_1, S_2, T_1 and T_2. This completes the proof.

Example 2.1. Let $\mathbb{R}$ be the real line with the usual norm $|.|$ and let $K = [-1,1]$. Define two mappings $S,T: K \to K$ by

$$T(x) = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0,1], \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1,0) \end{cases}$$

and

$$S(x) = \begin{cases} x, & \text{if } x \in [0,1], \\ -x, & \text{if } x \in [-1,0). \end{cases}$$

Then both S and T are asymptotically nonexpansive mappings with constant sequence $\{k_n\} = \{1\}$ for all $n \geq 1$ and uniformly L-Lipschtzian mappings with $L = \sup_{n\geq 1}\{k_n\}$ and hence they are nearly asymptotically nonexpansive mappings by Remark 1.1. Also the unique common fixed point of S and T, that is, $F = F(S) \cap F(T) = \{0\}$.

3. Conclusion

In this paper, we study hybrid mixed type iteration scheme for two nearly asymptotically nonexpansive self mappings and two asymptotically nonexpansive non-self mappings and establish some weak convergence theorems using the following conditions: (1) the space E has a Frěchet differentiable norm (2) dual space E^* of E has the Kadec-Klee (KK) property (3) the space E satisfies Opial's condition. Our results extend and generalize the corresponding results of [3]-[6] [11]-[17], and many others from the existing literature to the case of more general class of mappings and newly proposed hybrid mixed type iteration scheme considered in this paper.

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