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IMPROVED CONVERGENCE ANALYSIS FOR KING-WERNER-LIKE METHODS FREE OF DERIVATIVES USING RESTRICTED CONVERGENCE

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Abstract. We present a semilocal and local convergence analysis of some efficient King-Werner-like methods of order $1+\sqrt{2}$ free of derivatives in a Banach space setting. We use our idea of restricted domains, where the iterates lie leading to smaller Lipschitz constants yielding in turn to a more precise local as well as semilocal convergence analysis than in earlier studies. Numerical examples are presented to illustrate the theoretical results.

Keywords. Werner's method; Secant-type method; Banach space; Semilocal and local convergence analysis; Efficiency index.

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1. Introduction

In [6], Argyros and Ren studied King-Werner-like methods for approximating a locally unique solution $x^{\star}$ of equation

$$F(x) = 0, \tag{1}$$

where F is Fréchet-differentiable operator defined on a convex subset of a Banach space $\mathbb{B}_1$ with values in a Banach space $\mathbb{B}_2$.

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In particular, they studied the semilocal convergence analysis of method defined for $n = 0, 1, 2, \ldots$ by

$$(2) \qquad \begin{aligned} x_{n+1} &= x_n - A_n^{-1} F(x_n), \\ y_{n+1} &= x_{n+1} - A_n^{-1} F(x_{n+1}), \end{aligned}$$

where x_0, y_0 are initial points, $A_n = [x_n, y_n; F]$ and $[x, y; F]$ denotes a divided difference of order one for operator F at points $x, y \in \Omega$ [2,4,7] satisfying

$$(3) \qquad [x, y; F](x - y) = F(x) - F(y) \quad for\ each\ x, y \in \Omega\ with\ x \neq y.$$

If F is Fréchet-differentiable on Ω, then $F'(x) = [x, x; F]$ for each $x \in \Omega$. The local convergence analysis of method (2) was given in [9] in the special case when $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}$. The convergence order of method (2) was shown to be $1 + \sqrt{2}$. Using the idea of restricted convergence domains, we improve the applicability of method (2).

The paper is organized as follows: Section 2 contains the semilocal convergence analysis of method (2), and Section 3 contains the local convergence analysis of method (2). The numerical examples including favorable comparisons with earlier studies such as [6,9] are presented in the concluding Section 4.

2. Semilocal convergence of method (2)

For the semilocal convergence analysis of method (2) requires the following auxiliary result on majorizing sequences. The proof of these results can be found in see [6].

Lemma 2.1. *Let $L_0 > 0$, $L > 0$, $s_0 \geq 0$, $t_1 \geq 0$ be given parameters. Denote by α the only root in the interval $(0, 1)$ of polynomial p defined by*

$$(4) \qquad p(t) = L_0 t^3 + L_0 t^2 + 2Lt - 2L.$$

Suppose that

$$(5) \qquad 0 < \frac{L(t_1 + s_0)}{1 - L_0(t_1 + s_1 + s_0)} \leq \alpha \leq 1 - \frac{2L_0 t_1}{1 - L_0 s_0},$$

where

$$(6) \qquad s_1 = t_1 + L(t_1 + s_0)t_1.$$

Then, scalar sequence $\{t_n\}$ defined for each $n = 1, 2, \dots$ by

$$t_0 = 0, s_{n+1} = t_{n+1} + \frac{L(t_{n+1}-t_n+s_n-t_n)(t_{n+1}-t_n)}{1-L_0(t_n-t_0+s_n+s_0)}, \quad for\ each\ n = 1, 2, \dots, \tag{7}$$
$$t_{n+2} = t_{n+1} + \frac{L(t_{n+1}-t_n+s_n-t_n)(t_{n+1}-t_n)}{1-L_0(t_{n+1}-t_0+s_{n+1}+s_0)}, \quad for\ each\ n = 0, 1, 2, ...$$

is well defined, increasing, bounded above by

$$t^{\star\star} = \frac{t_1}{1-\alpha} \tag{8}$$

and converges to its unique least upper bound $t^\star$ which satisfies

$$t_1 \le t^\star \le t^{\star\star}. \tag{9}$$

Moreover, the following estimates hold

$$s_n - t_n \le \alpha(t_n - t_{n-1}) \le \alpha^n(t_1 - t_0), \tag{10}$$

$$t_{n+1} - t_n \le \alpha(t_n - t_{n-1}) \le \alpha^n(t_1 - t_0) \tag{11}$$

and

$$t_n \le s_n \tag{12}$$

for each $n = 1, 2, \dots$.

Denote by $U(w,\xi), \overline{U}(w,\xi)$, the open and closed balls in $\mathbb{B}_1$, respectively, with center $w \in \mathbb{B}_1$ and of radius $\xi > 0$. Next, we present the semilocal convergence of method (2) using $\{t_n\}$ as a majorizing sequence.

Theorem 2.2. *Let $F : \Omega \subset \mathbb{B}_1 \to \mathbb{B}_2$ be a Fréchet-differentiable operator. Suppose that there exists a divided differentiable $[.,.,;F]$ of order one for operator F on $\Omega \times \Omega$. Moreover, suppose that there exist $x_0, y_0 \in \Omega$, $L_0 > 0$, $L > 0$, $s_0 \ge 0$, $t_1 \ge 0$ such that*

$$A_0^{-1} \in L(\mathbb{B}_2, \mathbb{B}_1) \tag{13}$$

$$\|A_0^{-1}F(x_0)\| \le t_1, \tag{14}$$

$$\|x_0 - y_0\| \le s_0, \tag{15}$$

(16) $$\|A_0^{-1}([x,y;F]-A_0)\| \leq L_0(\|x-x_0\|+\|y-y_0\|), \text{ for each } x,y\in\Omega$$

(17) $$\begin{aligned}\|A_0^{-1}([x,y;F]-[z,v;F])\| &\leq L(\|x-z\|+\|y-v\|),\\ &\text{for each } x,y,z\in\Omega_1=\Omega\cap U(x_0,\frac{1}{2L_0})\end{aligned}$$

(18) $$\overline{U}(x_0,t^\star)\subseteq\Omega$$

and hypotheses of Lemma 2.1 hold, where $A_0=[x_0;y_0;F]$ *and* $t^\star$ *is given in Lemma 2.1. Then, sequence* $\{x_n\}$ *generated by method (2) is well defined, remains in* $\overline{U}(x_0,t^\star)$ *and converges to a unique solution* $x^\star\in\overline{U}(x_0,t^\star)$ *of equation* $F(x)=0$*. Moreover, the following estimates hold for each* $n=0,1,2,...$

(19) $$\|x_n-x^\star\|\leq t^\star-t_n.$$

Furthermore, if there exists $R>t^\star$ *such that*

(20) $$U(x_0,R)\subseteq\Omega$$

and

(21) $$L_0(t^\star+R+s_0)<1,$$

then, the point $x^\star$ *is the only solution of equation* $F(x)=0$ *in* $U(x_0,R)$*.*

Proof. Simply notice that the iterates remain in Ω_1 which is a more precise location than Ω used in [6], since $\Omega_1\subseteq\Omega$. Then, in view of this the proof follows from the corresponding one in [6].

□

Remark 2.3. *(a) The limit point* $t^\star$ *can be replaced by* $t^{\star\star}$ *given in closed form by (8) in Theorem 2.1.*

(b) In [6], Argyros and Ren used the stronger condition

$$\|A_0^{-1}([x,y;F]-[z,v;F])\|\leq L_1(\|x-z\|+\|y-v\|) \text{ for each } x,y,z,v\in\Omega.$$

Notice that from we have

$$L_0\leq L_1 \text{ and } L\leq L_1$$

holds in general and $\frac{L_1}{L_0}$ *can be arbitrarily large [2–6]. Moreover, it follows from the proof of Theorem 2.2 that hypothesis (17) is not needed to compute an upper bound for* $\|A_0^{-1}F(x_1)\|$. *Hence, we can define the more precise (than* $\{t_n\}$*) majorizing sequence* $\{\bar{t}_n\}$ *(for* $\{x_n\}$*) by*

$$\begin{aligned}&\bar{t}_0=0,\bar{t}_1=t_1,\bar{s}_0=s_0,\bar{s}_1=\bar{t}_1+L_0(\bar{t}_1+\bar{s}_0)\bar{t}_1,\\ &\bar{s}_{n+1}=\bar{t}_{n+1}+\frac{L(\bar{t}_{n+1}-\bar{t}_n+\bar{s}_n-\bar{t}_n)(\bar{t}_{n+1}-\bar{t}_n)}{1-L_0(\bar{t}_n-\bar{t}_0+\bar{s}_n+\bar{s}_0)} \quad for\ each\ n=1,2,...\end{aligned} \tag{22}$$

and

$$\bar{t}_{n+2}=\bar{t}_{n+1}+\frac{L(\bar{t}_{n+1}-\bar{t}_n+\bar{s}_n-\bar{t}_n)(\bar{t}_{n+1}-\bar{t}_n)}{1-L_0(\bar{t}_{n+1}-\bar{t}_0+\bar{s}_{n+1}+\bar{s}_0)} \quad for\ each\ n=0,1,... \tag{23}$$

Then, using a simple induction argument we have that

$$\bar{t}_n\le t_n, \tag{24}$$

$$\bar{s}_n\le s_n, \tag{25}$$

$$\bar{t}_{n+1}-\bar{t}_n\le t_{n+1}-t_n, \tag{26}$$

$$\bar{s}_n-\bar{t}_n\le s_n-t_n \tag{27}$$

and

$$\bar{t}^\star=\lim_{n\to\infty}\bar{t}_n\le t^\star. \tag{28}$$

Furthermore, if $L_0<L$*, then (24)-(27) are strict for* $n\ge 2$, $n\ge 1$, $n\ge 1$, $n\ge 1$, *respectively. Clearly, sequence* $\{\bar{t}_n\}$ *increasing converges to* $\bar{t}^\star$ *under the hypotheses of Lemma 2.1 and can replace* $\{t_n\}$ *as a majorizing sequence for* $\{x_n\}$ *in Theorem 2.2. Finally, the old sequences using* L_1 *instead of L in [6] are less precise than the new ones.*

3. **Local convergence of method (2)**

We present the local convergence of method (2) in this section. We have:

Theorem 3.1. *Let $F:\Omega\subseteq\mathbb{B}_1\to\mathbb{B}_2$ be a Fréchet-differentiable operator. Suppose that there exist $x^\star\in\Omega$, $l_0>0$ and $l>0$ such that for each $x,y,z,u\in\Omega$*

$$F(x^\star)=0,\ F'(x^\star)^{-1}\in L(\mathbb{B}_2,\mathbb{B}_1), \tag{29}$$

$$\|F'(x^\star)^{-1}([x,y;F]-F'(x^\star))\|\leq l_0(\|x-x^\star\|+\|y-x^\star\|) \text{ for each } x,y\in\Omega \tag{30}$$

$$\begin{aligned}\|F'(x^\star)^{-1}([x,y;F]-[z,u;F])\| &\leq l(\|x-z\|+\|y-u\|),\\ &\text{for each } x,y,z,u\in\Omega_2:=\Omega\cap U(x^*,\frac{1}{2l_0})\end{aligned} \tag{31}$$

and

$$\overline{U}(x^\star,\rho)\subseteq\Omega, \tag{32}$$

where

$$\rho=\frac{1}{(1+\sqrt{2})l+2l_0}. \tag{33}$$

Then, sequence $\{x_n\}$ generated by method (2) is well defined, remains in $\overline{U}(x^\star,\rho)$ and converges to $x^\star$ with order of $1+\sqrt{2}$ at least, provided that $x_0,y_0\in U(x^\star,\rho)$. Moreover, the following estimates

$$\|x_{n+2}-x^\star\|\leq\frac{\sqrt{2}-1}{\rho^2}\|x_{n+1}-x^\star\|^2\|x_n-x^\star\| \tag{34}$$

and

$$\|x_n-x^\star\|\leq(\frac{\sqrt{\sqrt{2}-1}}{\rho})^{F_n-1}\|x_1-x^\star\|^{F_n} \tag{35}$$

hold for each $n=1,2,\ldots$, where F_n is a generalized Fibonacci sequence defined by $F_1=F_2=1$ and $F_{n+2}=2F_{n+1}+F_n$.

Proof. As in the proof of Theorem 2.2.

□

Remark 3.2. *(a) For the special case $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}$, the radius of convergence ball for method (2) is given in [10] by*

$$\rho_\star = \frac{s^\star}{M}, \tag{36}$$

where $s^\star \approx 0.55279$ is a constant and $M > 0$ is the upper bound for $|F(x^\star)^{-1}F''(x)|$ in the given domain Ω. Using (31) we have

$$\|F'(x^\star)^{-1}(F'(x) - F'(y))\| \le 2l\|x - y\| \quad for\ any\ x, y \in \Omega. \tag{37}$$

That is, we can choose $l = \frac{M}{2}$. Simply set $l_0 = l$, we have from (33) that

$$\rho = \frac{2}{(3+\sqrt{2})M} = \frac{2(3-\sqrt{2})}{5M} \approx \frac{0.63432}{M} > \frac{s^\star}{M} = \rho_\star. \tag{38}$$

Therefore, even in this special case, a bigger radius of convergence ball for method (2) has been given in Theorem 3.1.

(b) Notice that we have

$$l_0 \le l_1 \text{ and } l \le l_1 \tag{39}$$

$$\|A_0^{-1}([x, y; F] - [z, u; F])\| \le l_1(\|x - z\| + \|y - u\|) \text{ for each } x, y, z, u \in \Omega.$$

The radius given in [6]:

$$\rho_0 = \frac{1}{(1+\sqrt{2})l_1 + l_0} \le \rho. \tag{40}$$

Moreover, if $l < l_1$, then $\rho_0 < \rho$ and the new error bounds(34) and (35) are tighter than the old ones in [6] using ρ_0 instead of ρ.

4. Numerical examples

We present some numerical examples in this section.

Example 4.1. *Let $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}$, $\Omega = (-1, 1)$ and define F on Ω by*

$$F(x) = e^x - 1. \tag{41}$$

Then, $x^\star = 0$ is a solution of Eq. (1.1), and $F'(x^\star) = 1$. Note that for any $x, y, z, u \in \Omega$, we have

$$
\begin{aligned}
&|F'(x^\star)^{-1}([x,y;F]-[z,u;F])|\\
&=|\int_0^1 (F'(tx+(1-t)y)-F'(tz+(1-t)u))dt|\\
&=|\int_0^1\int_0^1 (F''\big(\theta(tx+(1-t)y)+(1-\theta)(tz+(1-t)u)\big)\\
&\times\big(tx+(1-t)y-(tz+(1-t)u)\big)d\theta dt|\\
&=|\int_0^1\int_0^1 (e^{\theta(tx+(1-t)y)+(1-\theta)(tz+(1-t)u)}\big(tx+(1-t)y-(tz+(1-t)u)\big)d\theta dt|\\
&\le \int_0^1 e|t(x-z)+(1-t)(y-u)|dt\\
&\le \frac{e}{2}(|x-z|+|y-u|)
\end{aligned}
\tag{42}
$$

and

$$
\begin{aligned}
&|F'(x^\star)^{-1}([x,y;F]-[x^\star,x^\star;F])| = |\int_0^1 F'(tx+(1-t)y)dt - F'(x^\star)|\\
&=|\int_0^1 (e^{tx+(1-t)y}-1)dt|\\
&=|\int_0^1 (tx+(1-t)y)(1+\tfrac{tx+(1-t)y}{2!}+\tfrac{(tx+(1-t)y)^2}{3!}+\cdots)dt|\\
&\le|\int_0^1 (tx+(1-t)y)(1+\tfrac{1}{2!}+\tfrac{1}{3!}+\cdots)dt|\\
&\le\frac{e-1}{2}(|x-x^\star|+|y-x^\star|).
\end{aligned}
\tag{43}
$$

That is to say, the Lipschitz condition (31) and the center-Lipschitz condition (30) are true for $l_1 = \frac{e}{2}, l = \frac{e^{\frac{1}{e-1}}}{2}$ and $l_0 = \frac{e-1}{2}$, respectively. Using (33) in Theorem 3.1, we can deduce that the radius of convergence ball for method (2) is given by

$$\rho_0 = \frac{1}{(1+\sqrt{2})l_1+2l_0} = \frac{2}{(3+\sqrt{2})e-2} \approx 0.200018471, \tag{44}$$

which is smaller than the corresponding radius

$$\rho = \frac{1}{(1+\sqrt{2})l+2l_0} \approx 0.2578325131698342986 \tag{45}$$

Let us choose $x_0 = 0.2, y_0 = 0.199$. Suppose sequences $\{x_n\}$ and $\{y_n\}$ are generated by method (2). Table 1 gives a comparison results of error estimates for Example 4.1, which shows that tighter error estimates can be obtained from the new (34) or (35) using ρ instead of ρ_0 used in [6].

Hence the new results are more precise than the old ones in [6].

TABLE 1. The comparison results of error estimates for Example 4.1

n	using ρ new (35)	using ρ_0 old (35)
3	0.0498	0.0828
4	0.0031	0.0142
5	2.9778e-06	1.7305e-04
6	1.7108e-13	4.4047e-09
7	5.4309e-31	3.4761e-20

Example 4.2. *Let $\mathbb{B}_1 = \mathbb{B}_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$, equipped with the max norm and $\Omega = \overline{U}(0,1)$. Define function F on Ω, given by*

$$F(x)(s) = x(s) - 5\int_0^1 stx^3(t)dt, \tag{46}$$

and the divided difference of F is defined by

$$[x,y;F] = \int_0^1 F'(tx+(1-t)y)dt. \tag{47}$$

Then, we have

$$[F'(x)y](s) = y(s) - 15\int_0^1 stx^2(t)y(t)dt, \;\; for \;\; all \;\; y \in \Omega. \tag{48}$$

We have $x^\star(s) = 0$ for all $s \in [0,1]$, $l_0 = 3.75$ and $l = l_1 = 7.5$. Using Theorem 3.1, we can deduce that the radius of convergence ball for method (2) is given by

$$\rho_0 = \rho = \frac{1}{(1+\sqrt{2})l + 2l_0} \approx 0.039052429. \tag{49}$$

Example 4.3. *Let $\mathbb{B}_1 = \mathbb{B}_2 = C[0,1]$ be equipped with the max norm and $\Omega = U(0,r)$ for some $r > 1$. Define F on Ω by*

$$F(x)(s) = x(s) - y(s) - \mu\int_0^1 G(s,t)x^3(t)dt, \quad x \in C[0,1],\ s \in [0,1].$$

$y \in C[0,1]$ is given, μ is a real parameter and the Kernel G is the Green's function defined by

$$G(s,t) = \begin{cases} (1-s)t & \text{if } t \le s, \\ s(1-t) & \text{if } s \le t. \end{cases}$$

Then, the Fréchet derivative of F is defined by

$$(F'(x)(w))(s) = w(s) - 3\mu \int_0^1 G(s,t)x^2(t)w(t)dt, \quad w \in C[0,1],\ s \in [0,1].$$

Let us choose $x_0(s) = y_0(s) = y(s) = 1$ *and* $|\mu| < \frac{8}{3}$. *Then, we have that*

$$\|I - A_0\| \leq \tfrac{3}{8}\mu, \quad A_0^{-1} \in L(\mathbb{B}_2, \mathbb{B}_1),$$
$$\|A_0^{-1}\| \leq \tfrac{8}{8-3|\mu|}, \quad s_0 = 0, \quad t_1 = \tfrac{|\mu|}{8-3|\mu|}, \quad L_0 = \tfrac{3(1+r)|\mu|}{2(8-3|\mu|)},$$

and

$$L = \frac{3r|\mu|}{8-3|\mu|}.$$

Let us choose $r = 3$ *and* $\mu = \frac{1}{2}$. *Then, we have that*

$$t_1 = 0.076923077, \quad L_0 \approx 0.461538462, \quad L = L_1 \approx 0.692307692$$

and

$$\frac{L(t_1+s_0)}{1 - L_0(t_1+s_1+s_0)} \approx 0.057441746, \quad \alpha \approx 0.711345739,$$
$$1 - \frac{2L_0t_1}{1 - L_0s_0} \approx 0.928994083.$$

That is, condition (5) is satisfied and Theorem 2.2 applies.

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