Commun. Optim. Theory 2016 (2016), Article ID 9

MATHRES
Communications in Optimization Theory
Available online at http://cot.mathres.org

CHARACTERIZATION OF A CLASS OF COPOSITIVE MATRICES

MOURAD NAFFOUTI[1], ABDELJELIL BACCARI[2,*]

[1]Department of Mathematics, University of Tunis El Manar FST, Tunisia

[2]Department of Mathematics, University of Tunis ENSIT, Tunisia

Abstract. Testing a matrix on copositivity is a topic of recent interest and practical importance in optimization modeling. In this paper, we present a some constructive characterizations for a class of copositive matrices with certain spectral properties. The characterizations is based onto the orthogonal projection of the nonnegative orthant onto the subspace spanned by the eigenvectors of a given indefinite matrix. Moreover, we give a new necessary conditions for copositive matrices.

Keywords. Copositive matrices; Eigenvalues; Eigenvector; Nonnegative vector.

1. Introduction

Many problems in engineering, decision sciences, and operations research are formulated as nonlinear optimization problems. Solving nonlinear optimization problems remains a task for future work. A standard quadratic problems is a nonlinear optimization problems with nonlinear objective functions and linear constraints.

It is well known that, a given symmetric matrix is copositive if and only if the optimal value of the standard quadratic optimization problem is non-negative. We deduce that, there exists a relationship between the standard quadratic optimization problems and the copositivity. For further explanation see [1, 3].

*Corresponding author

E-mail addresses: mouradnaffouti@gmail.com (M. Naffouti), abdbaccari@gmail.com (A. Baccari)

Received March 18, 2014

The theory of copositive matrices was initiated by Motzkin [11] in 1952. The copositive matrices arises in various fields of applied mathematics. Many results have appeared in several directions: characterization of copositives matrices and extension to quadratic programming, see for example [2, 4, 5, 6, 7, 8, 9, 10, 13]. However, it is still difficult to determine whether a given matrix is copositive. A matrix is said to be copositive whenever its quadratic form is non-negative on the non-negative orthant. Clearly A is copositive if it is either positive semi-definite or it all its elements are nonnegative. Checking copositivity of a given matrix is a co-NP-complete problem, for further details, see [12]. In this paper, we study some subset of copositive matrices with certain spectral properties and use the orthogonal projection of an element of the nonnegative orthant onto the subspace spanned by the eigenvectors of a matrix A. If a nonnegative vector x in the subspace spanned by the eigenvectors corresponding to the nonnegative eigenvalues, then the matrix A is copositive. It was shown by C. R. Johnson and R. Reams (see [8]) that a copositive matrix A cannot have a nonnegative vector in the subspace spanned by the eigenvectors corresponding to the negative eigenvalues. We conclude that an element of the nonnegative orthant can be decomposed as the sum of an element of the subspace spanned by the eigenvectors corresponding to negative eigenvalues of A and an element of the subspace spanned by the eigenvectors corresponding to nonnegative eigenvalues of a matrix A.

The paper is divided into four part. First, we state the definition of copositive matrices. In Section 2, we introduce the notations and definitions. Section 3 is devoted to establishing the new characterization of copositives matrices, which based onto the orthogonal projection onto the subspace spanned by the eigenvectors of a given matrix A. Finally, some corollaries are furnished in section 4.

2. Preliminaries

Throughout this paper we will use the following notations and definitions.

- $\mathbb{R}^n_+$ is the non-negative orthant.
- $x \in \mathbb{R}^n_+$ i.e. $x_i \geq 0 \;\forall\; i = 1, 2, ..., n$.
- A is a real symmetric $n \times n$ matrix.

- It is well known that
$$A = \sum_{i=1}^{k} \lambda_i v_i v_i^T + \sum_{i=k+1}^{n} \lambda_i v_i v_i^T$$
where λ_i is the eigenvalue corresponding to eigenvector v_i of A.
- E_- is the subspace spanned by the eigenvectors $v_1, v_2, \ldots, v_k$ corresponding to negative eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ of A.
- E_+ is the subspace spanned by the eigenvectors $v_{k+1}, v_{k+2}, \ldots, v_n$ corresponding to non-negative eigenvalues $\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_n$ of A.
- $Tr(A) = \sum_{i=1}^{n} \lambda_i$, where λ_i the eigenvalues of A.
- $P_{E_-}(x)$ denotes the orthogonal projection of $\mathbb{R}^n_+$ onto the subspace E_-.
- $P_{E_+}(x)$ denotes the orthogonal projection of $\mathbb{R}^n_+$ onto the subspace E_+.
- $\|x\|$ denotes the Euclidiean norm of a vector x.

Definition 2.1. *A real symmetric $n \times n$ matrix A is called copositive iff:*

$$x^T A x \geq 0 \quad \textit{for all} \quad x \in \mathbb{R}^n_+,$$

i.e., if the quadratic form generated by A takes only non-negative values on the non-negative orthant $\mathbb{R}^n_+$.

Now, we present a relationship between the standard quadratic optimization problems and copositivity as follows:

- Sometimes one writes the standard quadratic optimization problems in the equivalent form
$$\min \{x^T A x : \ |x \in \mathbb{R}^n_+, x^T e e^T x = 1\},$$
where $e = [1, \ldots, 1]^T \in \mathbb{R}^n$ denoting a vector of ones.
- The corresponding dual problem is
$$\sup \{\lambda \in \mathbb{R} : A - \lambda \ ee^T \in COP_n\},$$
where COP_n is the cone of copositive matrices.

It is assumed in this paper that all matrices have a positive trace $Tr(A) > 0$.

3. Main results

In this section, we prove the main results of this paper.

Theorem 3.1. *Let A be an indefinite $n\times n$ symmetric matrix. If A is copositive then*

$$\|P_{E_-}(x)\|^2 \leq -\frac{\lambda_n}{\lambda_k}\|P_{E_+}(x)\|^2 \qquad \forall\, x\geq 0,$$

where λ_k,λ_n are the largest negative and nonnegative eigenvalues of A, respectively.

Proof. Let $\lambda_1,\lambda_2,\dots,\lambda_n$ be the eigenvalues of A. Suppose also that they are ordered as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k < 0$ and $0 \leq \lambda_{k+1} \leq \lambda_{k+2} \leq \cdots \leq \lambda_n$. Let $x\in\mathbb{R}^n_+$ such that $x=y+z$, where $y=P_{E_-}(x)$ and $z=P_{E_+}(x)$
We can write $x^TAx = y^TAy+z^TAz$. Since A is copositive, we have

$$y^TAy+z^TAz\geq 0,$$

which is equivalent to

$$y^TAy\geq -z^TAz.$$

Since the conditions, $y^TAy\leq\lambda_k\|y\|^2$ and $z^TAz\leq\lambda_n\|z\|^2$.

Hence,

$$\|y\|^2\leq -\frac{\lambda_n}{\lambda_k}\|z\|^2.$$

In Theorem 3.1, we have given a necessary condition for copositivity. Using this theorem and we assume that the matrix A has only two eigenvalues, we can now characterize copositivity, which is described in the following theorem.

Theorem 3.2. *Let A be an indefinite $n\times n$ symmetric matrix. If A has only two eigenvalues, λ_k,λ_n with $\lambda_k<0\leq\lambda_n$ then A is copositive if and only if*

$$\|P_{E_-}(x)\|^2 \leq -\frac{\lambda_n}{\lambda_k}\|P_{E_+}(x)\|^2 \qquad \forall\, x\geq 0,$$

where λ_k,λ_n are the largest negative and nonnegative eigenvalues of A, respectively.

Proof. Let $\lambda_1,\lambda_2,\dots,\lambda_n$ be the eigenvalues of A. Suppose also that they are ordered as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k < 0$ and $0 \leq \lambda_{k+1} \leq \lambda_{k+2} \leq \cdots \leq \lambda_n$ and $x\in\mathbb{R}^n_+$ such that $x=y+z$, where $y=P_{E_-}(x)$ and $z=P_{E_+}(x)$.

Suppose that

$$\|P_{E_-}(x)\|^2 \leq -\frac{\lambda_n}{\lambda_k}\|P_{E_+}(x)\|^2 \qquad \forall\, x\geq 0. \tag{3.1}$$

It is clear that

Combining this inequalities with (3.1), we obtain

$$x^T Ax \geq -\frac{\lambda_1 \lambda_n}{\lambda_k} \|z\|^2 + \lambda_{k+1} \|z\|^2.$$

We deduce that

$$x^T Ax \geq \frac{\lambda_k \lambda_{k+1} - \lambda_1 \lambda_n}{\lambda_k} \|z\|^2.$$

Since, $\lambda_1 = \lambda_k$ and $\lambda_{k+1} = \lambda_n$ then $\frac{\lambda_k \lambda_{k+1} - \lambda_1 \lambda_n}{\lambda_k} = 0$. Hence $x^T Ax \geq 0$. Finaly the matrix A is copositive.

We now use Theorem 3.2 to prove a weaker result, in Theorem 3.3

Theorem 3.3. *Let A be a real symmetric matrix of order n as follows*

$$A = \begin{pmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & A_r \end{pmatrix},$$

where $r < n$ and A_i has only two eigenvalues $\quad \forall\, i = 1, 2, \dots r$.

If A_i is copositive for all $i = 1, 2, \dots r$, then A is copositive.

Proof. Let $x = [y_1^T, y_2^T, \dots, \dots, y_r^T]^T \ \in \mathbb{R}^n_+$ and the matrix A_i is copositive for all $i = 1, 2, \dots r$, we have

$$x^T Ax = y_1^T A_1 y_1 + y_2^T A_2 y_2 + \cdots + y_r^T A_r y_r.$$

Hence, A is copositive.

4. Some corollaries

Corollary 4.1. *Let A be a real $n \times n$ symmetric matrix and x is an element of the nonnegative orthant. If $x \in E_+$, then A is positive semi-definite.*

Corollary 4.2. *Let A be an indefinite $n \times n$ symmetric matrix. If there exists $x \in \mathbb{R}^n_+$ such that*

$$\|P_{E_-}(x)\|^2 > -\frac{\lambda_n}{\lambda_k}\|P_{E_+}(x)\|^2$$

then A is not copositive, where λ_k, λ_n are the largest negative and nonnegative eigenvalues of A, respectively..

5. Conclusions

In this paper, we have been presented some properties for copositive matrices with certain spectral properties. Our approach presented here is based on the orthogonal projection of an element of the nonnegative orthant onto the subspace spanned by the eigenvectors of a given matrix A. This approach can provide a criterion to characterize some copositives matrices. In the general case, checking copositivity remains a difficult problem.

Acknowledgements

This project was funded by the Deanship of Scientific Research (No. 00/UR/15-04). The authors are grateful to the reviewers for useful suggestions which improved the contents of this article.

REFERENCES

[1] I. M. Bomze, Regularity versus degeneracy in dynamics, games, and optimization: a unified approach to different aspects. SIAM Rev. 44 (2002), 394–414.

[2] I. M. Bomze, Copositive optimization-recent developments and applications. European J. Oper. Res. 216 (2012), 509–520.

[3] S. Burer, On the copositive representation of binary and continuous nonconvex quadratic programs. Math. Program. 120 (2009), 479–495.

[4] M. Dür, Copositive Programming a Survey. Recent Advances in Optimization and its Applications in Engineering, Springer-Verlag, Berlin, Heidelberg. 2010.

[5] J. B. Hiriart-Urruty and A. Seeger, A variational approach to copositive matrices. SIAM Rev. 52 (2010), 593–629.

[6] A. J. Hoffman and F. Pereira, On copositive matrices with -1, 0, 1 entries. J. Comb. Number Theory 14 (1973), 302–309.

[7] B. Jargalsaikhan, Indefinite copositive matrices with exactly one positive eigenvalue or exactly one negative eigenvalue. Electron. J. Linear Algebra 26 (2013), 754–761.

[8] C. R. Johnson and R. Reams, Spectral theory of copositive matrices. Linear Algebra Appl. 395 (2005), 275–281.

[9] W. Kaplan, A test for copositive matrices. Linear Algebra Appl., 313 (2000), 203–206.

[10] P. Li and Y. Y. Feng, Criteria for copositive matrices of order four. Linear Algebra Appl. 194 (1993), 109–124.

[11] T. S. Motzkin, Copositive quadratic forms. National Bureau of Standards Report, 1818 (1952), 11–22.

[12] K. G. Murty and S. N. Kabadi, Some NP-Complete Problems in Quadratic and Linear Programming. Math. Program. 39 (1987), 117–129.

[13] H. Väliaho, Criteria for Copositive Matrices. Linear Algebra Appl. 81 (1986), 19–34.