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A STUDY FOR FLOW INTERACTIONS IN HETEROGENEOUS POROUS MEDIA

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Abstract. The central topic of this work is fluid flow through porous media. Therefore it is necessary to make a review on fluid dynamics.This is mainly done in order to give a good rational for Darcy's law, by taking into account the porosity and permeability of the media. Here we study a simplified system which describes a two phase flow (freshwater and saltwater) with a sharp interface in coastal areas. This kind of flow is modeled by a coupled system of an elliptic and a degenerate parabolic equation. Many authors have studied these systems and mostly have used regularisation techniques. Here we first write down the derived equations without regularisation, then we establish conditions to prove theoretical results on the existence and uniqueness of the solution.

Keywords. Porous media, Fluid dynamics.

1. Introduction

When dealing with fluids, it is assumed the continuum assumption. This assumption takes the matter as continuous. This, of course, is contrary to the physical reality, where as far as it is known, matter consists mainly of void. As long as the dimension of the control volume of the fluid is small enough compared to it bulk volume, but large enough so it contains a large number of molecules, the continuum assumption works fine. Once the continuum assumption in fluid mechanics has been accepted, the next step is to find a way to describe the motion of the fluid.

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When fresh water and salt water are in contact, an interface or transition zone occurs caused by the hydrodynamic dispersion. In these conditions, it is essential to model the extraction of seawater intrusion and locate freshwater-saltwater interface. For simplicity, it has been assumed that the two fluids are immiscible and the areas occupied by each fluid are separated by a sharp interface (for more details the reader is referred to [2] and [3]). This modeling approach does not describe the nature and behavior of the transition zone, but does give information on the motion of the salt water front. This information is important for the control of seawater intrusion and the optimum use of fresh groundwater.

To better study this very important phenomenon, one should look deeply for a mathematical model in order to have results and deduct predictive ideas and solutions. The first model was analyzed from all angles since W.B. Ghyben (1888) and B.H. Herzberg (1901) (see [9]) observed that in coastal zone, fresh water floats on the denser salt water and derived their simplest model describing the depth of the interface, assuming the immobility of the two fluids. In this situation the model is a coupled system consisting of an elliptic and a degenerate parabolic equations, we refer the reader to [3], [4] and [5] for more details.

2. Mathematical models

The system of equations for the flow of two immiscible fluid phases separated by a sharp interface, is given by the mass conservation and the Darcy's law in the form

$$S_i \frac{\partial(\Phi_i)}{\partial t} + \nabla.(v_i) = q_i \quad i = f, s, \tag{2.1}$$

$$v_i = -\frac{k\rho_i g}{\mu_i}\nabla\Phi_i, \tag{2.2}$$

where $i = f$ or s denotes the freshwater and saltwater phases; Φ_i, v_i, S_i, k, μ_i, ρ_i and g are the hydraulic head, the velocity, the water storativity's coefficient, the soil permeability, the dynamic viscosity, the density and the gravitational acceleration, respectively. The position of the interface can be determined by the following equation:

$$h = (1+\delta)\Phi_s - \delta\Phi_f, \tag{2.3}$$

where $\delta = \frac{\rho_f}{\rho_s - \rho_f}$, is the density contrast beween the two fluids with the salt water density $\rho_s = 1025\ kg/m^3$ and the fresh water density $\rho_f = 1000\ kg/m^3$.

2.1. Boundary and initial conditions

To complete this model, we add boundary and initial conditions such that

$$h = h_D, \qquad \text{on } \Gamma \times [0,T], \tag{2.4}$$

$$\Phi_f = \Phi_{f,D}, \qquad \text{on } \Gamma \times [0,T]. \tag{2.5}$$

The initial condition for the depth of the interface is given by

$$h(x,0) = h^0(x) \qquad \text{in } \Omega. \tag{2.6}$$

Many authors contributed in the study of such problems using different approaches; for example, in [8] the authors proved an existence result of solution using a technique developed by Alt and Luckauss [1]. The authors in [6] have proposed the global existence of solutions by using Schauder's theorem [6] combined with a parabolic regularization [8]. In this paper, we prove the existence and uniqueness of solution for a simplied model without regularization, adapting results of Gasmi et Al [7].

2.2. Simplified models

In the confined aquifer (Figure 1), it is assumed that the fresh water is quasi-static to give a degenerate elliptic-parabolic system.Therefore, system ((2.1) - (2.3)) is written in the form

$$n_e \frac{\partial h}{\partial t} - \operatorname{div}(K(x)T_s(h)\nabla h) + \operatorname{div}(K(x)T_s(h)\nabla \Phi_f) = -I_s \quad \text{in } \Omega, \tag{2.7}$$

$$-\operatorname{div}\left(K(x)\ H_2 \nabla \Phi_f\right) + \operatorname{div}\left(K(x)T_s(h)\nabla h\right) = I_s + I_f \quad \text{in } \Omega, \tag{2.8}$$

$$h = h_D, \quad \Phi_f = \Phi_{f,D} \quad \text{on } \Gamma \times [0,T], \tag{2.9}$$

$$h(x,0) = h_0(x) \quad \text{in } \Omega, \tag{2.10}$$

where H_2 is the aquifer thickness and $T_s(h) = H_2 - h$, the salt water zone thickness.

3. Existence and uniqueness of solutions

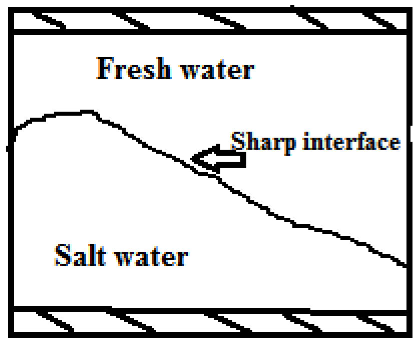

FIGURE 1. Confined Aquifer

Let Ω be a connected open set in $\mathbb{R}^d$ ($d = 2$ or 3), with a Lipschitz boundary Γ. To ensure the existence and the uniqueness of the weak solution, for our case, we start by setting the following hypotheses :

(1) $K(x) \in L^\infty(\Omega)$, such that

$$K_- \leq K(x) \leq K_+ \ \ a.e. \ in \ \Omega$$

with $K_- = \min(K(x)) \ and \ K_+ = \max(K(x))$

(2) $T_s(h) \in L^\infty(\Omega, (0,T))$, such that

$$T_- \leq T_s(h) \leq T_+ \ \ a.e. \ in \ \Omega \times (0,T)$$

with $T_- = \min(T_s\ (h)) \ and \ T_+ = \max(T_s\ (h))$

(3) $h \in L^\infty(\Omega, (0,T))$, such that

$$\delta \leq h(x,t) \leq H_2 \ a.e. in \ \Omega \times (0,T),$$

where δ is a small positive real parameter. We mean by $a.e.$ almost everywhere.

The weak formulation of problem $((2.7)-(2.10))$ is written as follows :

$$\begin{cases} \text{Find } (h,\Phi_f) \in V_{h_D} \times V_{f_D}, \\ \int_\Omega n_e \frac{\partial h}{\partial t}\, v_1 + \int_\Omega K(x) T_s(h) \nabla h\, \nabla v_1 - \int_\Omega K(x) T_s(h) \nabla \Phi_f \nabla v_1, \\ = -\int_\Omega I_s v_1, \quad \forall v_1 \in V_0, \\ \int_\Omega K(x)\, H_2 \nabla \Phi_f\, \nabla v_2 - \int_\Omega K(x) T_s(h) \nabla h\, \nabla v_2, \\ = \int_\Omega (I_s + I_f) v_2, \quad \forall v_2 \in V_0, \end{cases} \tag{3.1}$$

where $V_\varphi = \left\{ v \in H^1(\Omega), v = \varphi \quad \text{on } \Gamma \right\}$ with $H^1(\Omega)$ the standard Sobolev space.

Proposition 3.1. *Under the assumptions* $(1.-3.)$*, problem* (3.1) *has a unique solution* $\left(h,\Phi_f\right) \in V_{h_D} \times V_{\Phi_f,D}$..

Proof. To prove this result, we will adapt the results of Gasmi et Al [7] to our model. By using a variational formulation, system (3.1) can be written in a simplified form as :

$$\left(n_e \frac{\partial h}{\partial t}, v_1 \right) + (K(x) T_s(h) \nabla h, \nabla v_1) - (K(x) T_s(h) \nabla \Phi_f\, , \nabla v_1) = (-I_s, v_1)\,, \tag{3.2}$$

$$\left(K(x)\, H_2 \nabla \Phi_f,\, \nabla v_2\right) - (K(x) T_s(h) \nabla h\, , \nabla v_2) = \left(I_s + I_f, v_2\right). \tag{3.3}$$

If we take $\nabla v_1 = H_2$ and $\nabla v_2 = T_s(h)$, we get

$$\int_\Omega n_e \frac{\partial h}{\partial t} v_1 + \int_\Omega K(x) T_s(h) \left(H_2 - T_s(h)\right) \nabla h \; = \int_\Omega -I_s v_1 + (I_s + I_f) v_2.$$

We deduce that

$$n_e \frac{\partial h}{\partial t} v_1 + K(x) T_s(h)\; h\; \nabla h = -I_s v_1 + (I_s + I_f) v_2 \text{ a.e in } \Omega \tag{3.4}$$

in the distribution's sense. Then we introduce a new unknown W such that

$$W = K(x) T_s(h)\; h\; \nabla h,$$

to obtain

$$(*) \qquad n_e \frac{\partial h}{\partial t} v_1 + W = -I_s v_1 + (I_s + I_f) v_2,$$

and

$$(**) \qquad W = K(x) T_s(h)\; h\; \nabla h.$$

By using Lax-Milgram's theorem twice: first to determine h from $(**)$ assuming that W is known and then to determine W from $(*)$.

Let us suppose that the unknown $W \in L^2(\Omega,(0,T))$ is known and we try to find $h \in V_{h_D}(\Omega,(0,T))$ from equation $(**)$. We write

$$a(h,v) = l(v),$$

where

$$a(h,v) = (K(x)T_s(h)\; h\; \nabla h, v) \tag{3.5}$$

$$l(v) = (W, v) \tag{3.6}$$

This problem has a unique solution h in $V_{h_D}(\Omega,(0,T))$. Fist we prove the continuity and coercity of $a(.,.)$:

- **Continuity of** $a(,.,)$: we write

$$|a(h,v)| = \left| \int_\Omega K(x) T_s(h)\; h\; \nabla h\; v.dx \right|,$$

according to Cauchy-Schwarz inequality, we have

$$|a(h,v)| \leq \max |K(x)T_s(h)\; h| \left(\int_\Omega |\nabla h|^2 dx \right)^{1/2} \left(\int_\Omega |v|^2 dx \right)^{1/2},$$

and from conditions $(1. - 3.)$, we obtain

$$|a(h,v)| \leq K_+ . T_+ .\; H_2 . \|\nabla h\|_{L^2(\Omega,(0,T))} \|v\|_{L^2(\Omega,(0,T))} .$$

This implies

$$|a(h,v)| \leq C \|h\|_{V(\Omega)} \|v\|_{V(\Omega)},\; C = K_+ T_+ H_2.$$

- **Coercivity of** $a(,.,)$: we write

$$a(v,v)=\int_{\Omega}K(x)T_s(h)\;h\;\nabla v\;v.dx,$$

from the fact of the uniform ellipticity, we have

$$a(v,v)\geq \min|K(x)T_s(h)\;h|\,\varepsilon\int_{\Omega}v^2.dx,$$

and from conditions $(1.-3.)$, we obtain

$$a(v,v)\geq \varepsilon K_{-}\delta\,\|v\|^2_{L^2(\Omega,(0,T))}\geq \varepsilon K_{-}\delta\,\|v\|^2_{V(\Omega)}\,.$$

Hence

$$a(v,v)\geq \alpha\,\|v\|^2_{V(\Omega)}\,,\qquad \alpha=\varepsilon K_{-}\delta.$$

Now if we define $A\in \pounds(V,V^{'})$, where $V^{'}$ is the dual of V , as

$$a(h,v)=<Ah,v>, \tag{3.7}$$

we can write

$$Ah=W\quad \text{a.e in } \Omega\times(0,T)\,. \tag{3.8}$$

Let us now demonstrate that A is an isomorphism from V into $V^{'}$. By applying the Riesz representation's theorem on the bilinear form $a(.,.)$, there exists $Ah\in V^{'}$such that

$$a(h,v)=<Ah,v>,\quad \forall v\in V.$$

Furthermore, we have

$$\|Av\|\leq c\,\|v\|\,,$$

thus A is continous and we have

$$<Av,v>\geq \alpha\,\|v\|^2\,.$$

Hence

$$\|Av\|\geq \alpha\,\|v\|\,.$$

Let C be a closed and convex subset of V and let $h\in C$, we can write

$$a(h,v-h)\geq l(v-h),\ \ \forall v\in C,$$

so that

$$< Ah, v-h > \geq < f, v-h >, \quad \forall v \in C. \tag{3.9}$$

If $\rho > 0$, then (3.9) is equivalent to finding $h \in C$ such that

$$< \rho\, f - \rho\, Ah + h - h, v - h > \leq 0, \quad \forall v \in C.$$

If we denote the projection operator by P_C , and use the projection theorem on the closed convex C, we get

$$h = P_C(\rho\, f - \rho\, Ah + h).$$

Hence we are now looking for a fixed point of the continuous map $F : C \to C$,lying in C that is defined by

$$F(v) = P_C(\rho\, f - \rho\, Av + v).$$

Now, if $v_1, v_2 \in C$, we have

$$\|F(v_1) - F(v_2)\| = \|P_C(\rho\, f - \rho\, Av_1 + v_1) - P_C(\rho\, f - \rho\, Av_2 + v_2)\|.$$

As the projection operator P_C is Lipschitzian, we have

$$\|F(v_1) - F(v_2)\| \leq \|v_1 - v_2 - \rho\, A(v_1 - v_2)\|.$$

Hence,

$$\begin{aligned} \|F(v_1) - F(v_2)\|^2 &\leq \|v_1 - v_2\|^2 - 2\rho < v_1 - v_2, A(v_1 - v_2) > + \rho^2 \| A(v_1 - v_2)\|^2 \\ &\leq (1 - 2\alpha\rho + \rho^2 c^2) \|v_1 - v_2\|^2. \end{aligned}$$

If we choose ρ such that $0 < \rho < \frac{2\alpha}{c^2}$ (c and α are the continuity and coercivity constants , respectively), then,

$$1 - 2\alpha\rho + \rho^2 c^2 < 1.$$

This leads to

$$\|F(v_1) - F(v_2)\| \leq \|v_1 - v_2\|.$$

Therefore F is a contraction and has a unique fixed point h in $C \subset V$. Let us now study equation $(*)$:

$$n_e \frac{\partial h}{\partial t} v_1 + W = -I_s v_1 + (I_s + I_f) v_2.$$

From (3.8), we get

$$h = A^{-1}W,$$

if we multiply equation $(*)$ by a test function $v \in V$ and integrate over Ω, we get

$$\int_\Omega \frac{\partial}{\partial t}(A^{-1}W)\, v - \int_\Omega \frac{W\, v}{n_e} = \int_\Omega \frac{(-I_s v_1 + (I_s + I_f)v_2)\, v}{n_e}\ .$$

If we define the norm $((.,.))$ on Ω by $((W,v)) = \int_\Omega A^{-1}(W)v$, we have

$$\frac{d}{dt}((W,v)) - \int_\Omega \frac{Wv}{n_e} = \int_\Omega \frac{(-I_s v_1 + (I_s + I_f)v_2)\, v}{n_e}\ , \tag{3.10}$$

$$W(0) = W_0.$$

This problem has a unique solution W in $L^2(\Omega,(0,T)) \cap L^\infty((0,T),V')$.

4. Conclusion

In this paper, we prove a theorical result for the existence and the uniqueness of the solution of the simplified system describing the evolution of the interface for the seawater intrusion problem in a confined aquifer. This is proved by decoupling the equations and using twice Lax-Milgram's theorem, first to determine the depth of the interface h from equation $(*)$, assuming that W ($W = K(x).T_s(h).h.\nabla h$) in equation $(**)$, is known, and then obtain W from equation $(**)$. It has to be pointed out that the system studied here has not been subject to any regularisation and the conditions (1. - 3.) used have been derived for the need to show the continuity and the coercivity of the bilinear form in the variational formulation (3.1) associated to our problem.

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