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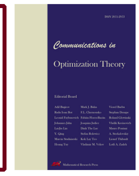

SOME SUZUKI TYPE FIXED POINT THEOREMS IN COMPLETE CONE b-METRIC SPACES OVER A SOLID VECTOR SPACE

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Abstract. In this paper, we prove some new Suzuki type fixed point theorems in complete cone b-metric spaces over a solid vector space. Our result complements recent results announced by many others.

Keywords. Fixed point; Cone b-metric space; Solid vector space.

1. Introduction and preliminaries

In this paper, we study some new Suzuki type fixed point theorems in complete cone $b-$metric spaces over solid vector spaces. Cone metric spaces have a long history (see Collatz [7], Du and E. Karapnar [10], Kumam et al. [19], Zabrejko [36], Janković et al. [18], Proinov [22], P.P. Zabrejko [36] and references therein). A unified theory of cone metric spaces over a solid vector space was developed in a recent paper of Proinov [22]. Recall that an ordered vector space with convergence structure $(Y, \preceq)$ is called a solid vector space if it can be endowed with a strict vector ordering $(\prec)$. The metric spaces with vector valued metric are known under various names such as pseudometric spaces [7], K-metric spaces [12, 36, 20], generalized metric spaces [30, 14], vector-valued metric spaces [3], cone-valued metric spaces [5, 6], cone metric spaces [15, 18].

In 2007, Huang and Zhang [15] introduced cone metric spaces as a generalization of metric spaces and extended the Banach contraction principle to cone metric spaces over a normal solid cone , being unaware that the cone metric spaces are known in the literature under various names, in particular, "K-metric space" comes from Russian and means in English exactly Cone metric space because "Cone" in Russian is "Konus". In the general theory of cone metric spaces (see [36] and [22]) the cone metric takes values in an ordered space with a convergence structure. In other words, Huang and Zhang consider only a special convergence structure on a solid vector space.

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Moreover, in the literature the convergence via interior points of the cone is one of the oldest (see for example, [5]). After them many authors have studied cone metric spaces over solid cone and fixed point theorems in such spaces (see [4, 8, 2, 18, 9, 23, 25, 27, 22, 28, 26, 29, 32, 33] and the references therein). The first paper is a survey of the paper on fixed point theorems in cone metric spaces obtained in 2007-2011. The second one gives a theory of cone metric spaces over a solid vector space.

In [17], Hussain and Shah introduced cone b -metric spaces as a generalization of b -metric spaces and cone metric spaces. They established some topological properties in such spaces and improved some recent results about KKM mappings in the setting of a cone b -metric space. Very recently, several papers have dealt with the cone b -metric space structure and fixed point theory or the variational principle for single-valued and multi-valued operators in cone b -metric spaces (see [1, 13, 16, 31] and the references therein).

In this paper, motivated and inspired by ideas of some recent papers such as Kadelburg et.al, [9], Edelstein [11], Piri and Marasi [21] Rezapour Hamlbarani [29], Suzuki [34] and Wong [35], , we obtain some fixed point theorems of contractive mappings in cone b-metric spaces. The results greatly generalize and improve the work of Kadelburg et al. [9], Huang and Zhang [15], Huang and Xu [16], Rezapour Hamlbarani [29] and many others.

Definition 1.1. [22] Let Y be a real vector space and S be the set of all infinite sequences in Y. A binary relation between S and Y is called a convergence on Y if it satisfies the following axioms:

$(C1)$ If $x_n \to x$ and $y_n \to y$, then $x_n + y_n \to x + y$.
$(C2)$ If $x_n \to x$ and $\lambda \in \mathbb{R}$, then $\lambda x_n \to \lambda x$.
$(C3)$ If $\lambda_n \to \lambda$ in $\mathbb{R}$ and $x \in Y$, then $\lambda_n x \to \lambda x$.

The pair $(Y, \to)$ is said to be a vector space with convergence. If $x_n \to x$, then $\{x_n\}$ is said to be a convergent sequence in Y, and the vector x is said to be a limit of $\{x_n\}$.

Definition 1.2. [22] Let $(Y, \to)$ be a vector space with convergence. An ordering $\preceq$ on Y is said to be a vector ordering if it is compatible with the algebraic and convergence structures on Y in the sense that the following are true:

$(V1)$ If $x \preceq y$, then $x + z \preceq y + z$, for every $z \in Y$.
$(V2)$ If $\lambda \geq 0$ and $x \preceq y$, then $\lambda x \preceq \lambda y$.
$(V3)$ If $x_n \to x$, $y_n \to y$, $x_n \preceq y_n$ for all n, then $x \preceq y$.

A vector space with convergence $(Y, \to)$ equipped with a vector ordering $\preceq$ is called an ordered vector space with convergence and is denoted by $(Y, \preceq, \to)$.

Definition 1.3. [22] Let $(Y, \preceq, \to)$ be an ordered vector space with convergence. A strict ordering $\prec$ on Y is said to be a strict vector ordering if it is compatible with the vector ordering, the algebraic structure and the convergence structure on Y in the sense that the following are true:

$(S1)$ If $x \prec y$, then $x \preceq y$.

$(S2)$ If $x \preceq y$ and $y \prec z$, then $x \prec z$.

$(S3)$ If $x \prec y$, then $x+z \prec y+z$, for every $z \in Y$.

$(S4)$ If $\lambda > 0$ and $x \prec y$, then $\lambda x \prec \lambda y$.

$(S5)$ If $x_n \to x$, $y_n \to y$ and $x \prec y$, then $x_n \prec y_n$ for all but finitely many n.

An ordered vector space with convergence $(Y, \preceq, \to)$ equipped with a strict vector ordering $\prec$ is denoted by $(Y, \preceq, \prec, \to)$.

Lemma 1.4. [22] *Strict vector ordering $\prec$ on a ordered vector space with convergence $(Y, \preceq, \to)$ satisfies also the following properties:*

$(S6)$ *If $\lambda < 0$ and $x \prec y$, then $\lambda x \succ \lambda y$.*

$(S7)$ *If $\lambda < \mu$ and $x \succ 0$, then $\lambda x \prec \mu x$.*

$(S8)$ *If $\lambda < \mu$ and $x \prec 0$, then $\lambda x \succ \mu x$.*

$(S9)$ *If $x \prec y$ and $y \preceq z$, then $x \prec z$.*

$(S10)$ *If $x \preceq y$ and $u \prec v$, then $x+u \prec y+v$.*

$(S11)$ *If $x \prec c$ for each $c \succ 0$, then $x \prec 0$.*

It turns out that an ordered vector space can be endowed with at most one strict vector ordering (see Proinov [22, Theore 5.1]).

Definition 1.5. [22] (Solid vector space) An ordered vector space with convergence endowed with a strict vector ordering is said to be a solid vector space.

Definition 1.6. (Cone b-metric space) Let X be a nonempty set, $s \geq 1$ be a given real number and let $(Y, \preceq, \to)$ be an ordered vector space with convergence. A vector-valued function $d \colon X \times X \to Y$ is said to be a cone metric on Y if the following conditions hold:

(d_1) $d(x,y) \succeq 0$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;

(d_2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(d_3) $d(x,y) \preceq s[d(x,z) + d(z,y)]$ for all $x, y, z \in X$.

The pair (X,d) is called a cone $b-$metric space with coefficient s over Y.

Let (X,d) be a cone $b-$metric space with coefficient s over a solid vector space $(Y, \preceq, \to)$. Then a sequence $\{x_n\}$ of points in X converges to $x \in X$ if and only if for every vector $c \in Y$ with $c \succ 0$, $d(x_n, x) \prec c$ for all but finitely many n. Recall also that a sequence $\{x_n\}$ in X is called a Cauchy sequence if for every $c \in Y$ with $c \succ 0$, there is $N \in \mathbb{N}$ such that $d(x_n, x_m) \prec c$ for all $n, m > N$. A cone metric space X is called complete if each Cauchy sequence in X is convergent.

By the careful analysis of the proof of Theorem 9.9 of [22], we obtain the following theorem. Because its proof is much similar to the proof of Theorem 9.9 of [22], we omit its proof.

Theorem 1.7. *Let (X,d) be a complete cone $b-$metric space with coefficient s over a solid vector space $(Y,\preceq,\prec,\rightarrow)$. Suppose $\{x_n\}$ is a sequence in X satisfying*

$$d(x_n,x_m)\preceq b_n,\ \ for\ all\ \ n,m\in\mathbb{N}\ \ with\ \ m\geq n, \tag{1}$$

where $\{b_n\}$ is a sequence in Y which converges to 0. Then $\{x_n\}$ converges to a point $x\in X$ with error estimate $d(x_n,x)\preceq sb_n$ for all $n\in\mathbb{N}$.

Definition 1.8. Let $(Y,\preceq,\prec,\rightarrow)$ be a solid vector space and $x,y\in Y$. We say that $x\succeq y$ if $x\prec y$ not established.

2. Main results

Theorem 2.1. *Let s be a real number such that $s\geq 1$ and let T be a self mapping on a complete cone $b-$metric space (X,d) with coefficient s over a normed solid vector space $(Y,\preceq,\prec,\rightarrow)$. Let k be a real number in $[0,1)$ and suppose that there exist functions α_i, $i=1,2,3,4,5$ of Y into $[0,\infty)$ such that*

- (A) $\alpha_1(t)+2\alpha_2(t)+\alpha_3(t)+\alpha_4(t)+2\alpha_5(t)\leq\frac{k}{s}\parallel t\parallel$, *for all* $t\in Y$,
- (B) *for any distinct $x,y\in X$, $\frac{1}{2s}d(x,Tx)\prec d(x,y)$ implies that*

$$\begin{aligned}\parallel d(x,y)\parallel d(Tx,Ty)\preceq{}&\alpha_1(d(x,y))d(x,y)+\alpha_2(d(x,y))d(x,Ty)+\alpha_3(d(x,y))d(Tx,y)\\&+\alpha_4(d(x,y))d(x,Tx)+\alpha_5(d(x,y))d(y,Ty).\end{aligned} \tag{2}$$

Then T has a unique fixed point.

Proof. Chose $x_0\in X$. Set $x_n=T^nx_0$ for every $n\in\mathbb{N}$. If there exists $n\in\mathbb{N}$ such that $x_n=Tx_n$ the proof is complete. So we assume that

$$d(x_n,Tx_n)\succ 0,\quad\forall n\in\mathbb{N}, \tag{3}$$

therefore,

$$\frac{1}{2s}d(x_n,Tx_n)\prec d(x_n,Tx_n),\quad\forall n\in\mathbb{N}. \tag{4}$$

From assumption of theorem, we have,

$$\begin{aligned}\| d(x_n,Tx_n) \| d(Tx_n,T^2x_n) &\preceq \alpha_1(d(x_n,Tx_n))d(x_n,Tx_n)+\alpha_2(d(x_n,Tx_n))d(x_n,T^2x_n)\\ &\quad+\alpha_3(d(x_n,Tx_n))d(Tx_n,Tx_n)+\alpha_4(d(x_n,Tx_n))d(x_n,Tx_n)\\ &\quad+\alpha_5(d(x_n,Tx_n))d(Tx_n,T^2x_n)\\ &\preceq \alpha_1(d(x_n,Tx_n))d(x_n,Tx_n)+s\alpha_2(d(x_n,Tx_n))d(x_n,Tx_n)\\ &\quad+s\alpha_2(d(x_n,Tx_n))d(Tx_n,T^2x_n)+\alpha_4(d(x_n,Tx_n))d(x_n,Tx_n)\\ &\quad+\alpha_5(d(x_n,Tx_n))d(Tx_n,T^2x_n),\end{aligned}$$

which implies that

$$d(Tx_n,T^2x_n) \preceq \frac{\alpha_1(d(x_n,Tx_n)+s\alpha_2(d(x_n,Tx_n))+\alpha_4(d(x_n,Tx_n))}{\| d(x_n,Tx_n) \| - s\alpha_2(d(x_n,Tx_n))-\alpha_5(d(x_n,Tx_n))} d(x_n,Tx_n).$$

From A, we get

$$d(Tx_n,T^2x_n)=d(x_{n+1},Tx_{n+1}) \preceq kd(x_n,Tx_n). \tag{5}$$

Therefore,

$$\begin{aligned}d(x_n,x_{n+1}) &\preceq kd(x_{n-1},x_{n-2})\\ &\preceq k^2d(x_{n-2},x_{n-3})\\ &\preceq k^3d(x_{n-3},x_{n-4})\\ &\vdots\\ &\preceq k^nd(x_0,x_1). \end{aligned}\tag{6}$$

On the other hand for $m>n$, we have

$$\begin{aligned}d(x_m,x_n) &= d(x_n,x_m)\\ &\preceq s[d(x_n,x_{n+1})+d(x_{n+1},x_m)]\\ &\preceq sd(x_n,x_{n+1})+s^2[d(x_{n+1},x_{n+2})+d(x_{n+2},x_m)]\\ &\preceq sd(x_n,x_{n+1})+s^2d(x_{n+1},x_{n+2})+s^3[d(x_{n+2},x_{n+3})+d(x_{n+3},x_m)]\\ &\preceq sd(x_n,x_{n+1})+s^2d(x_{n+1},x_{n+2})+\cdots+s^{m-n}d(x_{m-1},x_m).\end{aligned}\tag{7}$$

From (6) and (7), we obtain

$$\begin{aligned} d(x_m,x_n) &\preceq [sk^n + s^2k^{n+1} + s^3k^{n+2} + \cdots + s^{m-n-1}k^{m-2} + s^{m-n}k^{m-1}]d(x_0,x_1) \\ &= sk^n[1 + sk + (sk)^2 + (sk)^3 + \cdots + (sk)^{m-n-2} + (sk)^{m-n-1}]d(x_0,x_1) \\ &= sk^n[\frac{1-(sk)^{m-n}}{1-sk}]d(x_0,x_1) \\ &\preceq sk^n[\frac{1}{1-sk}]d(x_0,x_1). \end{aligned} \tag{8}$$

Since $k^n[\frac{1}{1-sk}] \to 0$ in $\mathbb{R}$, we find from $(C3)$ and Theorem 1.7 that there exists a point $x \in X$ such that

$$\lim_{n\to\infty} x_n = x \quad and \quad d(x_n,x) < sk^n[\frac{1}{1-sk}]d(x_0,x_1), \quad \forall n \in \mathbb{N}. \tag{9}$$

Now, we claim that

$$\begin{aligned} &(I) \quad \frac{1}{2s}d(x_n,Tx_n) \prec d(x_n,x), \quad \forall n \in \mathbb{N}, \\ &\qquad\qquad or \\ &(II) \quad \frac{1}{2s}d(Tx_n,T^2x_n) \prec d(Tx_n,x), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{10}$$

Arguing by contradiction, we assume that there exists $n \in \mathbb{N}$ such that

$$\frac{1}{2s}d(x_n,Tx_n) \succeq d(x_n,x) \; and \; \frac{1}{2s}d(Tx_n,T^2x_n) \succeq d(Tx_n,x). \tag{11}$$

Using (5) and (11), we have

$$\begin{aligned} d(x_n,Tx_n) &\preceq s[d(x_n,x) + d(x,Tx_n)] \\ &\preceq s[\frac{1}{2s}d(x_n,Tx_n) + \frac{1}{2s}d(Tx_n,T^2x_n)] \\ &\preceq \frac{1}{2}d(x_n,Tx_n) + \frac{1}{2}kd(x_n,Tx_n). \end{aligned}$$

Therefore $\frac{1-k}{2}d(x_n,Tx_n)\preceq 0$. Since $k\in[0,1)$, we get $d(x_n,Tx_n)\preceq 0$, which is contradiction with (3). From part (I) of (10) and (B), we get

$$\begin{aligned}\| d(x_n,x)\| d(Tx_n,Tx) &\preceq \alpha_1(d(x_n,x))d(x_n,x)+\alpha_2(d(x_n,x))d(x_n,Tx)\\ &\quad+\alpha_3(d(x_n,x))d(Tx_n,x)+\alpha_4(d(x_n,x))d(x_n,Tx_n)\\ &\quad+\alpha_5(d(x_n,x))d(x,Tx)\\ &\preceq \alpha_1(d(x_n,x))d(x_n,x)+s\alpha_2(d(x_n,x))d(x_n,Tx_n)\\ &\quad+s\alpha_2(d(x_n,x))d(Tx_n,Tx)+\alpha_3(d(x_n,x))d(Tx_n,x)\\ &\quad+s\alpha_4(d(x_n,x))d(x_n,Tx_n)+s\alpha_5(d(x_n,x))d(x,Tx_n)\\ &\quad+s\alpha_5(d(x_n,x))d(Tx_n,Tx),\end{aligned}$$

which implies that

$$\begin{aligned}d(Tx_n,Tx)\preceq&\ \frac{\alpha_1(d(x_n,x))}{\| d(x_n,x)\| -s\alpha_2(d(x_n,x))-s\alpha_5(d(x_n,x)}d(x_n,x)\\ &+\frac{s\alpha_2(d(x_n,x))+\alpha_4(d(x_n,x))}{\| d(x_n,x)\| -s\alpha_2(d(x_n,x))-s\alpha_5(d(x_n,x))}d(x_n,Tx_n)\\ &+\frac{\alpha_3(d(x_n,x))+s\alpha_5(d(x_n,x))}{\| d(x_n,x)\| -s\alpha_2(d(x_n,x))-s\alpha_5(d(x_n,x))}d(x,Tx_n).\end{aligned}$$

So, from (A), we get

$$\begin{aligned}d(Tx_n,Tx)&\preceq kd(x_n,x)+kd(x_n,Tx_n)+kd(x,Tx_n)\\ &\preceq kd(x_n,x)+ks[d(x_n,x)+d(x,Tx_n)]+kd(x,Tx_n)\\ &=(s+1)kd(x_n,x)+(s+1)kd(x,Tx_n)\\ &=(s+1)kd(x_n,x)+(s+1)kd(x,x_{n+1}).\end{aligned}\tag{12}$$

Let $0\prec c$ be arbitrary. Since $\lim_{n\to\infty}x_n=x$, we find there exists $N_1\in\mathbb{N}$ such that

$$d(x_n,x)\prec\frac{1}{2(s+1)k}c,\quad\forall n>N_1.\tag{13}$$

It follows from (12) and (13) that

$$d(Tx_n,Tx)\prec(s+1)k\frac{1}{2(s+1)k}c+(s+1)k\frac{1}{2(s+1)k}c=c,\quad\forall n>N_1.\tag{14}$$

So, from (13) and (14), for $n>N_1$, we get

$$\begin{aligned}d(x,Tx)&\preceq s[d(x,Tx_n)+d(Tx_n,Tx)]\\ &\preceq s\frac{1}{2(s+1)k}c+sc\\ &=s(\frac{1}{2(s+1)k}+1)c\end{aligned}$$

Since $0 \prec c$ is arbitrary, we get $x = Tx$. From part (II) of (10) and (B), we have

$$\begin{aligned}\| d(x_n,x) \| d(T^2x_n,Tx) &\preceq \alpha_1(d(Tx_n,x))d(Tx_n,x)+\alpha_2(d(Tx_n,x))d(Tx_n,Tx)\\ &\quad +\alpha_3(d(Tx_n,x))d(T^2x_n,x)+\alpha_4(d(Tx_n,x))d(Tx_n,T^2x_n)\\ &\quad +\alpha_5(d(Tx_n,x))d(x,Tx)\\ &\preceq \alpha_1(d(Tx_n,x))d(Tx_n,x)+s\alpha_2(d(Tx_n,x))d(Tx_n,T^2x_n)\\ &\quad +s\alpha_2(d(Tx_n,x))d(T^2x_n,Tx)+\alpha_3(d(Tx_n,x))d(T^2x_n,x)\\ &\quad +\alpha_4(d(Tx_n,x))d(Tx_n,T^2x_n)+s\alpha_5(d(Tx_n,x))d(x,T^2x_n)\\ &\quad +s\alpha_5(d(Tx_n,x))d(T^2x_n,Tx),\end{aligned}$$

which implies that

$$\begin{aligned}d(T^2x_n,Tx) &\preceq \frac{\alpha_1(d(Tx_n,x))}{\| d(Tx_n,x) \| -s\alpha_2(d(Tx_n,x))-s\alpha_5(d(Tx_n,x)}d(Tx_n,x)\\ &\quad +\frac{s\alpha_2(d(Tx_n,x))+\alpha_4(d(Tx_n,x))}{\| d(Tx_n,x) \| -s\alpha_2(d(Tx_n,x))-s\alpha_5(d(Tx_n,x))}d(Tx_n,T^2x_n)\\ &\quad +\frac{\alpha_3(d(Tx_n,x))+s\alpha_5(d(Tx_n,x))}{\| d(Tx_n,x) \| -s\alpha_2(d(Tx_n,x))-s\alpha_5(d(Tx_n,x))}d(x,T^2x_n). \qquad (15)\end{aligned}$$

From (15) and (A), we get

$$\begin{aligned}d(T^2x_n,Tx) &\preceq kd(Tx_n,x)+kd(Tx_n,T^2x_n)+kd(x,T^2x_n)\\ &\preceq kd(Tx_n,x)+ks[d(Tx_n,x)+d(x,T^2x_n)]+kd(x,T^2x_n)\\ &\preceq ksd(Tx_n,x)+ks[d(Tx_n,x)+d(x,T^2x_n)]+ksd(x,T^2x_n)\\ &= (1+s)kd(x_{n+1},x)+(1+s)kd(x_{n+2},x). \qquad (16)\end{aligned}$$

Using (13) and (16), we obtain

$$d(T^2x_n,Tx) \preceq (1+s)k\frac{1}{2(1+s)k}c+(1+s)k\frac{1}{2(1+s)k}c = c, \quad \forall n > N_1. \qquad (17)$$

Therefore, from (13) and (17), for $n > N_1$, we obtain

$$\begin{aligned}d(x,Tx) &\preceq s[d(x,T^2x_n)+d(T^2x_n,Tx)]\\ &= sd(x,x_{n+2})+sd(T^2x_n,Tx)\\ &\preceq s\frac{1}{2(1+s)k}c+sc\\ &= s(\frac{1}{2(1+s)k}+1)c.\end{aligned}$$

Since $0 \prec c$ is arbitrary, we get $x = Tx$.

Suppose that y is another fixed point of T such that $x \neq y$. Therefore

$$\frac{1}{2s} d(x,Tx) \prec d(x,y),$$

so from (B), we get

$$\begin{aligned}
\| d(x,y) \| d(x,y) = \| d(x,y) \| d(Tx,Ty) &\preceq \alpha_1(d(x,y))d(x,y) + \alpha_2(d(x,y))d(x,Ty) \\
&\quad + \alpha_3(d(x,y))d(Tx,y) + \alpha_4(d(x,y))d(x,Tx) \\
&\quad + \alpha_5(d(x,y))d(y,Ty) \\
&= \alpha_1(d(x,y))d(x,y) + \alpha_2(d(x,y))d(x,y) \\
&\quad + \alpha_3(d(x,y))d(x,y) + \alpha_4(d(x,y))d(x,x) \\
&\quad + \alpha_5(d(x,y))d(y,y) \\
&= [\alpha_1(d(x,y)) + \alpha_2(d(x,y)) + \alpha_3(d(x,y)]d(x,y) \\
&\preceq \frac{k}{s} \| d(x,y) \| d(x,y).
\end{aligned}$$

Therefore, $(1-\frac{k}{s}) \| d(x,y) \| d(x,y) \preceq 0$. Since $0 \leq (1-\frac{k}{s}) \| d(x,y) \|$, we get $d(x,y) \preceq 0$. But $d(x,y) \succ 0$. Therefore $x = y$.

Remark 2.2. Theorem extends Theorem 3.8 of [9], Theorem 1, Theorem 3 and Theorem 4 of [15], Theorem 2.1 and Theorem 2.7 of [16] and Theorem 2.3, Theorem 2.6, Theorem 2.7 and Theorem 2.8 of [29].

Example 2.3. Let $s = 1$ and $k = 0.99999$. Let $Y = C^1_{\mathbb{R}}[0,1]$ endowed with the norm $\| f \| = \| f \|_\infty + \| f' \|_\infty$. Define the pointwise ordering $\preceq$ and $\prec$ on Y by means of

$$x \preceq y \ \ if \ \ and \ \ only \ \ if \ \ x(t) \leq y(t) \ \ for \ \ each \ \ t \in [0,1],$$

$$x \prec y \ \ if \ \ and \ \ only \ \ if \ \ x(t) < y(t) \ \ for \ \ each \ \ t \in [0,1].$$

Then $(Y, \preceq, \prec, \rightarrow)$ is a normed solid vector space. Let $X = \{0\} \cup [2.5,3]$ and $d : X \times X \to Y$ defined by $d(x,y) = f_{x,y}$ where $f_{x,y}(t) = | x - y | e^t$. We define the self-mapping $T : X \to X$ as follows:

$$Tx = \begin{cases} 2.5, & x \in \{0\} \cup [2.5,3), \\ 2.6, & x = 3. \end{cases}$$

For $i = 1,2,3,4,5$, we define the mappings $\alpha_i : Y \to \mathbb{R}$ as follows:

$$\alpha_1(f) = \frac{9}{1000} \| f \|,\ \alpha_2(f) = \frac{9}{200} \| f \|,\ \alpha_3(f) = \frac{9}{20} \| f \|,$$
$$\alpha_4(f) = \frac{9}{20} \| f \|, \quad \alpha_5(f) = \frac{9}{20000} \| f \| .$$

Obviously, we have

$$\alpha_1(f)+2\alpha_2(f)+\alpha_3(f)+\alpha_4(f)+2\alpha_5(f)\leq \frac{k}{s}\parallel f\parallel .$$

We now consider the inequality

$$\frac{1}{2s}d(x,Tx)=\frac{1}{2}d(x,Tx)\prec d(x,y), \tag{18}$$

when $x\neq y$, and the inequality

$$\begin{aligned}\parallel d(x,y)\parallel d(Tx,Ty)\preceq \alpha_1(d(x,y))d(x,y)+\alpha_2(d(x,y))d(x,Ty)+\alpha_3(d(x,y))d(Tx,y)\\ +\alpha_4(d(x,y))d(x,Tx)+\alpha_5(d(x,y))d(y,Ty)\end{aligned} \tag{19}$$

for those $x,y\in X$ (with $x\neq y$) which satisfy (18).

Case 1**:** For $x=0$ and $y=3$, we have

$$\frac{1}{2}f_{0,T0}(t)=\frac{1}{2}f_{0,2.5}(t)=2.5e^t<3e^t=f_{0,3}(t),\ \ for\ \ each\ \ t\in[0,1].$$

Hence, (18) holds. Also we observe that

$$\begin{aligned}&\alpha_1(f_{0,3})f_{0,3}(t)+\alpha_2(f_{0,3})f_{0,2.6}(t)+\alpha_3(f_{0,3})f_{2.5,3}(t)+\alpha_4(f_{0,3})f_{0,2.5}(t)+\alpha_5(f_{0,3})f_{3,2.6}(t)\\ &=\parallel f_{0,3}\parallel [\frac{9}{1000}\times 3+\frac{9}{200}\times 2.6+\frac{9}{20}\times 0.5+\frac{9}{20}\times 2.5+\frac{9}{20000}\times 0.4]e^t\\ &=\parallel f_{0,3}\parallel [\frac{27}{1000}+\frac{234}{200}+\frac{45}{200}+\frac{225}{200}+\frac{36}{200000}]e^t\\ &=\parallel f_{0,3}\parallel \frac{5400+234000+45000+225000+36}{200000}e^t\\ &=\parallel f_{0,3}\parallel \frac{509436}{20000}e^t>\parallel f_{0,3}\parallel \frac{1}{10}e^t=\parallel f_{0,3}\parallel f_{2.5,2.6}(t)=\parallel f_{0,3}\parallel f_{T0,T3}(t).\end{aligned}$$

Hence, we have

$$\begin{aligned}\parallel d(0,3)\parallel d(T0,T3)\prec \alpha_1(d(0,3))d(0,3)+\alpha_2(d(0,3))d(0,T3)+\alpha_3(d(0,3))d(T0,3)\\ +\alpha_4(d(0,3))d(0,T3)+\alpha_5(d(0,3))d(3,T3).\end{aligned}$$

So, (19) holds.

Case 2**:** For $x=3$ and $y=0$, we have we have

$$\frac{1}{2}f_{3,T3}(t)=\frac{1}{2}f_{3,2.6}(t)=0.4e^t<3e^t=f_{3,0}(t),\ \ for\ \ each\ \ t\in[0,1].$$

Hence, (18) holds. Also we observe that

$$\begin{aligned}
&\alpha_1(f_{3,0})f_{3,0}(t)+\alpha_2(f_{3,0})f_{3,2.5}(t)+\alpha_3(f_{2.5,0})f_{2.6,0}(t)+\alpha_4(f_{3,0})f_{3,2.5}(t)+\alpha_5(f_{3,0})f_{0,2.5}(t)\\
&\qquad=\| f_{3,0} \| [\frac{9}{1000}\times 3+\frac{9}{200}\times 2.5+\frac{9}{20}\times 2.6+\frac{9}{20}\times 0.5+\frac{9}{20000}\times 2.5]e^t\\
&\qquad=\| f_{3,0} \| [\frac{27}{1000}+\frac{225}{2000}+\frac{234}{200}+\frac{45}{200}+\frac{225}{200000}]e^t\\
&\qquad=\| f_{3,0} \| \frac{5400+22500+234000+45000+225}{200000}e^t=\| f_{3,0} \| \frac{307125}{200000}e^t\\
&\qquad>\| f_{3,0} \| 0.1e^t=\| f_{3,0} \| f_{2.6,2.5}(t)=\| f_{3,0} \| f_{T3,T0}(t).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\| d(3,0) \| d(T3,T0) \prec{}& \alpha_1(d(3,0))d(3,0)+\alpha_2(d(3,0))d(3,T0)+\alpha_3(d(3,0))d(T3,0)\\
&+\alpha_4(d(3,0))d(3,T0)+\alpha_5(d(3,0))d(0,T0).
\end{aligned}$$

So, (19) holds.

Case 3**:** For $x\in[2.5,3)$ and $y=0$, we have

$$\frac{1}{2}f_{x,Tx}(t)=\frac{1}{2}\mid x-2.5\mid e^t<2.5e^t\leq xe^t=f_{x,0}(t),\ \ for\ each\ t\in[0,1].$$

Hence, (18) holds. Also we observe that

$$\begin{aligned}
&\alpha_1(f_{x,0})f_{x,0}(t)+\alpha_2(f_{x,0})f_{x,2.5}(t)+\alpha_3(f_{x,0})f_{2.5,0}(t)+\alpha_4(f_{x,0})f_{x,2.5}(t)+\alpha_5(f_{x,0})f_{0,2.5}(t)\\
&\qquad=\| f_{x,0} \| [\frac{9}{1000}\times x+\frac{9}{200}\times\mid x-2.5\mid+\frac{9}{20}\times 2.5+\frac{9}{20}\times\mid x-0.5\mid+\frac{9}{20000}\times 2.5]e^t\\
&\qquad\geq\| f_{x,0} \| [\frac{9}{1000}\times\frac{25}{10}+0+\frac{9}{20}\times 2.5+0+\frac{9}{20000}\times\frac{25}{10}]e^t\\
&\qquad=\| f_{x,0} \| [\frac{225}{10000}+\frac{225}{200}+\frac{225}{200000}]e^t\\
&\qquad=\| f_{x,0} \| \frac{2250+225000+225}{200000}e^t\\
&\qquad=\| f_{x,0} \| \frac{227475}{200000}e^t\\
&\qquad>0=\| f_{x,0} \| f_{2.5,2.5}=\| f_{x,0} \| f_{Tx,T0}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\| d(x,0) \| d(Tx,T0) \prec{}& \alpha_1(d(x,0))d(x,0)+\alpha_2(d(x,0))d(x,T0)+\alpha_3(d(x,0))d(Tx,0)\\
&+\alpha_4(d(x,0))d(x,Tx)+\alpha_5(d(x,0))d(0,T0).
\end{aligned}$$

So, (19) holds.

Case 4**:** For $x=0$ and $y\in[2.5,3)$, note (18) is clear. Also, (19) holds (similar to Case 3).

Case 5**:** For $x = 3$ and $y \in [2.5, 3)$, if $y \in [2.5, 2.8)$, we have

$$\frac{1}{2} f_{3,T3}(t) = \frac{4}{20} e^t <| 3 - y | e^t = f_{3,y}(t), \;\; for \;\; each \;\; t \in [0,1].$$

Hence, (18) holds. Also we observe that

$$\begin{aligned}
&\alpha_1(f_{3,y})f_{3,y}(t) + \alpha_2(f_{3,y})f_{3,2.5}(t) + \alpha_3(f_{3,y})f_{2.6,y}(t) + \alpha_4(f_{3,y})f_{3,2.6}(t) + \alpha_5(f_{3,y})f_{y,2.5}(t) \\
&\quad = \| f_{3,y} \| [\frac{9}{1000} \times | 3 - y | + \frac{9}{200} \times 0.5 + \frac{9}{20} \times | y - 2.6 | + \frac{9}{20} \times 0.4 + \frac{9}{20000} \times | y - 2.5 |] e^t \\
&\quad \geq \| f_{3,y} \| [\frac{45}{2000} + \frac{36}{200}] e^t \\
&\quad > \| f_{3,y} \| \frac{1}{10} e^t = \| f_{3,y} \| f_{T3,Ty}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\| d(3,y) \| d(T3,Ty) \prec \alpha_1(d(3,y))d(3,y) + \alpha_2(d(3,y))d(3,Ty) + \alpha_3(d(3,y))d(Tx,y) \\
+ \alpha_4(d(T3,3))d(x,Tx) + \alpha_5(d(3,y))d(y,Ty).
\end{aligned}$$

So, (19) holds. If $y \in [2.8, 3)$ then (18) is not true so we do not need to check (19).

Case 6**:** For $x \in [2.5, 3)$ and $y = 3$, if $x \in [2.5, 2.83)$, we have

$$\begin{aligned}
\frac{1}{2} f_{x,Tx}(t) - f_{x,y}(t) &= (\frac{1}{2} \, | x - 2.5 | - | 3 - x |) e^t \\
&= (\frac{3}{2} x - 4.25) e^t \\
&< (\frac{3}{2} 2.8 - 4.25) e^t \\
&= (4.2 - 4.25) e^t < 0, \;\; for \;\; each \;\; t \in [0,1].
\end{aligned}$$

Hence, (18) holds. Also we observe that

$$\begin{aligned}
&\alpha_1(f_{x,3})f_{x,3}(t) + \alpha_2(f_{x,3})f_{x,2.6}(t) + \alpha_3(f_{x,3})f_{2.6,3}(t) + \alpha_4(f_{x,3})f_{x,2.5}(t) + \alpha_5(f_{x,3})f_{3,2.6}(t) \\
&\quad = \| f_{x,3} \| [\frac{9}{1000} \times | x - 3 | + \frac{9}{200} \times | x - 2.6 | + \frac{9}{20} \times 0.4 + \frac{9}{20} \times | x - 0.5 | + \frac{9}{20000} \times 0.4] e^t \\
&\quad \geq \| f_{x,3} \| [\frac{36}{200} + \frac{36}{200000}] e^t \\
&\quad = \| f_{x,3} \| \frac{36000 + 36}{200000} e^t \\
&\quad = \| f_{x,3} \| \frac{36036}{200000} e^t \\
&\quad > \| f_{x,3} \| \times 0.1 = \| f_{x,3} \| f_{2.5,2.6} = \| f_{x,0} \| f_{Tx,T3}.
\end{aligned}$$

Hence, we have

$$\| d(x,3) \| d(Tx,T3) \prec \alpha_1(d(x,3))d(x,3) + \alpha_2(d(x,3))d(x,T3) + \alpha_3(d(x,3))d(Tx,3) + \alpha_4(d(x,3))d(x,Tx) + \alpha_5(d(x,3))d(3,T3).$$

So, (19) holds. If $x \in [2.8,3)$ then (18) is not true so we do not need to check (19). Hence T satisfies in assumption of Theorem 2.1 and 2.5 is the unique fixed point of T.

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