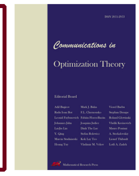
Commun. Optim. Theory 2016 (2016), Article ID 3

MATHRES
Communications in Optimization Theory
Available online at http://cot.mathres.org



ON THE EXISTENCE OF SOLUTIONS OF QUASI-EQUILIBRIUM PROBLEMS (UPQEP), (LPQEP), (UWQEP) AND (LWQEP) AND RELATED PROBLEMS

TRAN VAN SU[1,*], THAN VAN DINH[2]

[1]Department of Mathematics, Quangnam University, 102 Hung Vuong, Tam Ky, Vietnam

[2]Department of of Natural Sciences, Colleges of Binhphuoc, Tan Binh, Dong Xoai, Binh Phuoc, Vietnam

Abstract. The purpose of the present paper is to prove two main theorems concerning C-monotone and C-convex multivalued mappings where C is given cone and its applications to the quasi-equilibrium problems (UPQEP), (LPQEP), (UWQEP) and (LWQEP). Moreover, we also derive some sufficient conditions on the existence of solutions of the general vector α optimization problems $(GVOP)_\alpha$ and derive a sufficient condition on the existence of equilibrium points.

Keywords. Upper Pareto quasi-equilibrium problems; Lower Pareto quasi-equilibrium problems; Upper Weakly quasi-equilibrium problems; Lower Weakly quasi-equilibrium problems.

1. Introduction

Let X and Y be real topological vector spaces and Y with partial order S generated by a convex cone C. Let us denote by $x - y \in C$ instead of xSy for every $x, y \in Y$. Let M be a nonempty subset in Y, we define efficient sets of M with respect to C in different senses cases as $PMin(M|C)$, $IMin(M|C)$, $WMin(M|C)$ and $PrMin(M|C)$ (see Luc, D.T [7]). If $\bar{x} \in M$ is an element of efficient sets $\alpha Min(M|C|)$ (where $\alpha \in \{P, I, W, Pr\}$), then point $\bar{x}$ is called an α efficient point of M with respect to C. For instances, when $\alpha = I$, $\bar{x}$: Ideal efficient point; $\alpha = P$, $\bar{x}$: Pareto efficient point; $\alpha = W$, $\bar{x}$: Weak efficient point and $\alpha = Pr$, $\bar{x}$: Proper efficient

*Corresponding author.

E-mail addresses: tranuu63@gmail.com (T.V. Su), dinhanalysis@gmail.com (T.V. Dinh)

Received November 11, 2015

point, etc. Let D be a nonempty subset in X, denote by 2^D indicates the family of all subsets of D. Let the multivalued mapping $F: D \longrightarrow 2^Y$. The general vector α optimization problem corresponding to D, F and C for $\alpha \in \{I, P, W, Pr\}$denoted by $(GVOP)_\alpha$, by means of finding $\bar{x} \in D$ such that

$$(GVOP)_\alpha \qquad\qquad F(\bar{x}) \cap \alpha Min(F(D)|C) \neq \emptyset.$$

A point $\bar{x}$ solved $(GVOP)_\alpha$ is called a solution of $(GVOP)_\alpha$ and a point $\bar{y} \in \alpha Min(F(D)|C)$ is called α optimal value of $(GVOP)_\alpha$.

Let X, Y and Z be topological vector spaces with D and K be nonempty subsets in X and Z, respectively and let C be a cone in Y. Let us consider the multivalued mappings S, T, F, G and H, where $S: D \longrightarrow 2^D$, $T: D \times D \longrightarrow 2^K$, $F, G, H: K \times D \times D \longrightarrow 2^Y$. From now on, unless otherwise specify, we always suppose that G and H are two different multivalued mappings and let F of the form $F(y, x, x') = G(y, x', x) - H(y, x, x')$ for all $(y, x, x') \in K \times D \times D$. In the present paper we shall deal with some problems related as follows:

Problem 1.1. [18] (UPQEP), Upper Pareto quasi-equilibrium problem. Find $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \text{ and}$$

$$F(y, \bar{x}, x) := G(y, x, \bar{x}) - H(y, \bar{x}, x) \not\subset -(C \setminus l(C)), \text{ for all } x \in S(\bar{x}),\ y \in T(\bar{x}, x).$$

Problem 1.2. [18] (LPQEP), Lower Pareto quasi-equilibrium problem. Find $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \text{ and}$$

$$(G(y, x, \bar{x}) - H(y, \bar{x}, x)) \cap -(C \setminus l(C)) = \emptyset, \text{ for all } x \in S(\bar{x}),\ y \in T(\bar{x}, x).$$

Problem 1.3. [18] (UWQEP), Upper weakly quasi-equilibrium problem. Find $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \text{ and}$$

$$F(y, \bar{x}, x) := G(y, x, \bar{x}) - H(y, \bar{x}, x) \not\subset -int(C), \text{ for all } x \in S(\bar{x}),\ y \in T(\bar{x}, x).$$

Problem 1.4. [18] (LWQEP), Lower weakly quasi-equilibrium problem. Find $\bar{x} \in D$ such that

$$\bar{x} \in S(\bar{x}) \text{ and}$$

$$(G(y,x,\bar{x}) - H(y,\bar{x},x)) \cap -(intC) = \emptyset, \text{ for all } x \in S(\bar{x}),\ y \in T(\bar{x},x).$$

The above problems are called γ quasi-equilibrium problems involving D,K,S,T,F with respect to C, where γ is respectively one of the following qualifications: Upper Pareto, Lower Pareto, Upper weakly and Lower weakly. The above problems are proposed by Lin and Tan [18] in which the existence of solutions are derived. The related problems has been studied by many other authors (see, e.g., Ansari [1], Chang and Pang [5], Luc and Tan [8], Minh and Tan [12], Tan [17], etc and the references therein).

The remainder of this paper is organized as follows. After some preliminaries and definitions, two main theorems for quasi-equilibrium problems concerning multivalued mappings in Hausdorff locally convex topological linear spaces are well-presented analysis in Section 3. As an application, we provide some sufficient conditions on the existence of solutions of general vector α optimization problems, where $\alpha \in \{I,P,Pr,W\}$ and the existence of vector equilibrium points is also derived.

2. Preliminaries and definitions

In this subsection, let X,Y,D and F be given as in section 1. The effective domain of F is denoted as $domF = \{x \in D \,|\, F(x) \neq \emptyset\}$. We recall some definitions as follows:

Definition 2.1. [5, 6-8, 18] Let $F : D \longrightarrow 2^Y$ be a multivalued mapping

(i) F is said to be upper C-continuous at $\bar{x} \in domF$ if for all neighborhood V of the origin in Y there is a neighborhood U of $\bar{x}$ such that

$$F(x) \subset F(\bar{x}) + V + C$$

holds for all $x \in U \cap domF$.

(ii) F is said to be upper C-continuous on D if F is upper C-continuous at any point of $domF$.

(iii) F is said to be lower C-continuous at $\bar{x} \in domF$ if for all neighborhood V of the origin in Y there is a neighborhood U of $\bar{x}$ such that

$$F(\bar{x}) \subset F(x) + V - C$$

holds for all $x \in U \cap domF$.

(iv) F is said to be lower C-continuous on D if F is lower C-continuous at any point of $domF$.

(v) F is said to be $C-$ continuous on D if F is simultaneously upper C-continuous and lower C-continuous on D.

(vi) F is said to be $C-$ convex if D is convex and for any $x, y \in D$, any $t \in [0,1]$ we have

$$tF(x) + (1-t)F(y) \subset F(tx + (1-t)y) + C.$$

(vii) F is said to be $C-$ concave if D is convex and for any $x, y \in D$, any $t \in [0,1]$ we have

$$tF(x) + (1-t)F(y) \subset F(tx + (1-t)y) - C.$$

Definition 2.2. [7] Let M be a nonempty subset of Y. We say that

(i) $x \in M$ is an ideal efficient (or ideal minimal) point of M with respect to C if $M \subset x + C$. The set of ideal minimal points of M is denoted by $IMin(M|C)$.

(ii) $x \in M$ is an efficient (or Pareto minimal or nondominated) point of M with respect to C if $M \cap (x - C) \subset x + C \cap (-C)$. The set of efficient points of M is denoted by $PMin(M|C)$.

(iii) $x \in M$ is a (global) proper efficient point of M with respect to C if there exists a convex cone $\widetilde{C}$ which is not the whole space and contains $C \setminus C \cap (-C)$ in its interior such that $x \in PMin(M|\widetilde{C})$. The set of proper efficient points of M is denoted by $PrMin(M|C)$.

(iv) Supposing that $intC \neq \emptyset$, point $x \in M$ is a weak efficient point of M with respect to C if $x \in PMin(M|intC \cup \{0\})$. The set of weak efficient points of M is denoted by $WMin(M|C)$.

Following Luc, T. D [7, Proposition 2.2] that

$$PrMin(M|C) \subset PMin(M|C) \subset WMin(M|C)$$

and moreover $IMin(M|C) = PMin(M|C)$ if $IMin(M|C) \neq \emptyset$.

3. Main results

From now on, unless otherwise specify let us always assume that X,Y and Z be Hausdorff locally convex topological vector spaces with D and K be nonempty compact convex subsets in X and Z, respectively and cone C pointed closed convex with its interior nonempty in Y. The multivalued mappings S,T,F,G,H are given as in section 2 with nonempty closed values.

3.1. Two main theorems for quasi-equilibrium problems concerning multivalued mappings and some problems related

Definition 3.1.1. Let $F : D\times D \longrightarrow 2^Y$ be a multivalued mapping with nonempty values. We say that F is upper C-monotone* if $F(x,y)+F(y,x)\subset -C$ holds for any $x,y\in D$.

Remark 3.1.2. If F is a vector valued function from $D\times D$ into Y and upper C-monotone* simultaneously, then F is upper $C-$ monotone. In fact, by definition for every $(x,y)\in D\times D$, we get the inclusion of $F(x,y)+F(y,x)\subset -C$ holds, which is equivalent to $F(x,y)\subset -F(y,x)-C$. Consequently, F is upper C-monotone. Note that Definition 3.1.1 is new in this paper to us. From there we derive two main theorems concerning multivalued mappings as follows:

Theorem 3.1.3. *Let $G,H : D\times D \longrightarrow 2^Y$ be multivalued mappings with nonempty values. Assume that all the following conditions are fulfilled*

(A) $G(x,x)=\{0\}$ and $H(x,x)=\{0\}$ for all $x\in D$;

(B) G is upper C-monotone;*

(C) For all $x\in D$, $x=\sum\limits_{i\in I}\lambda_i x_i$, where $x_i\in D, \lambda_i\in[0;1]$ for all $i\in I$, $\sum\limits_{i\in I}\lambda_i=1$, I is finite index set,

$$G(y,x)\subset\sum_{i\in I}\lambda_i G(y,x_i)-C, \text{ for all } y\in D.$$

(D) For all $x\in D$, the multivalued mapping $H(x,\ .\ ) : D\longrightarrow 2^Y$ is C-convex;

(E) For all $y\in D$, the set $A(y):=\{x\in D : G(y,x)-H(x,y)\subset intC\}$ is open in D.

Then there exists a point $\bar{x}\in D$ such that

$$G(y,\bar{x})-H(\bar{x},y)\not\subset intC, \text{ for all } y\in D.$$

Suppose, moreover, that Y has a countable neighborhood base and there exists a nonempty compact convex subset $B\subset Y$ does not contain zero such that $C=\{tb\,|\,b\in B,\ t\geq 0\}$ and for *every $c \in C \setminus \{0\}$, there exist unique $b \in B$ and $t > 0$ such that $c = tb$. Then there exists at least one point $\bar{x} \in D$ such that*

$$G(y,\bar{x}) - H(\bar{x},y) \not\subset C \setminus \{0\}, \textit{ for all } y \in D.$$

To prove the theorem we need the following proposition

Proposition 3.1.4. *Let Y be Hausdorff locally convex topological vector space and let C be a cone in Y. Suppose, moreover, that there exists a compact convex subset $B \subset Y$ does not contain zero such that $C = \{tb \,|\, b \in B,\ t \geq 0\}$ and for every $c \in C \setminus \{0\}$, there exist unique $b \in B$ and $t > 0$ such that $c = tb$. Then, if Y has a countable neighborhood base then there exists a pointed closed convex cone, say $\widetilde{C}$, such that $int(\widetilde{C}) \neq \emptyset$ and $C \setminus \{0\} \subset int(\widetilde{C})$.*

Proof. We denote by Y' instead of the topological dual space of Y. Since B does not contain zero in Y, one can separate $\{0\}$ and B by a nonzero vector $\xi \in Y'$ such that $\varepsilon := inf\{\xi(b) \,|\, b \in B\} > 0$. By taking $S = \{b \in Y \,|\, \xi(b) > 0\}$ and $V = \{x \in Y \,|\, |\xi(x)| \leq \frac{\varepsilon}{2}\}$. It is easy to see that V is a neighborhood of the origin in Y such that $B + V \subset S$. By choosing

$$\widetilde{C} = \{t(b+v) \mid b \in B, v \in V, t \geq 0\},$$

then $\widetilde{C}$ is pointed closed convex cone in Y and $C \setminus \{0\} \subset int(\widetilde{C})$, which completes the proof.

Proof of Theorem 3.1.3. We first prove that there exists a point $\bar{x} \in D$ such that

$$G(y,\bar{x}) - H(\bar{x},y) \not\subset intC, \text{ for all } y \in D.$$

For any fixed $y \in D$, we put

$$S(y) := \{x \in D : G(y,x) - H(x,y) \not\subset intC\}.$$

Obviously, $S(y)$ is nonempty subset in D for all $y \in D$. Furthermore, $S(y)$ is a closed subset in D because $S(y) = D \cap X \setminus A(y)$. We next prove that

$$\bigcap_{y \in D} S(y) \neq \emptyset.$$

In fact, we can consider $\{y_i : i \in I\}$ is a finite arbitrary subset in D. Thus, for every $z \in conv\{y_i : i \in I\}$, we have a representation as follows

$$z = \sum_{i\in I} \lambda_i y_i, \text{ where } \lambda_i \in [0;1] \forall i \in I, \sum_{i\in I} \lambda_i = 1.$$

From there we conclude that

$$z \in \bigcup_{i\in I} S(y_i).$$

In fact, posit to the contrary that

$$z \notin \bigcup_{i\in I} S(y_i).$$

Therefore $z \notin S(y_i)$ for all $i \in I$. It follows from the definition that

$$G(y_i, z) - H(z, y_i) \subset int(C), \text{ for all } i \in I.$$

Consequently

$$\sum_{i\in I} \lambda_i \Big(G(y_i, z) - H(z, y_i) \Big) \subset \sum_{i\in I} \lambda_i int(C) \subset int(C). \tag{1}$$

We invoke the hypotheses of (B) to deduce that the multivalued mapping G is upper $C-$ monotone, which is equivalent to

$$G(y_i, y_j) + G(y_j, y_i) \subset -C, \text{ for all } i, j \in I.$$

This together with the hypotheses (C), it leads to

$$\begin{aligned}\sum_{i\in I} \lambda_i G(y_i, z) &= \sum_{i\in I} \lambda_i G(y_i, \sum_{j\in I} \lambda_j y_j) \subset \sum_{i,j\in I} \lambda_i \lambda_j G(y_i, y_j) - C \\ &= \frac{1}{2} \sum_{i,j\in I} \lambda_i \lambda_j \Big(G(y_i, y_j) + G(y_j, y_i) \Big) - C \\ &\subset - \sum_{i,j\in I} \lambda_i \lambda_j C - C \subset -C - C \subset -C.\end{aligned} \tag{2}$$

On the other hand, from (A) and (D) it follows that

$$\begin{aligned}\sum_{i\in I} \lambda_i H(z, y_i) &\subset H(z, \sum_{i\in D} \lambda_i y_i) + C \\ &= H(z, z) + C = \{0\} + C = C\end{aligned} \tag{3}$$

Combining (2) and (3), yields that

$$\sum_{i\in I} \lambda_i \Big(G(y_i, z) - H(z, y_i) \Big) \subset -C. \tag{4}$$

Since C is pointed cone, and this combines with both (1) and (4), we have a contradiction. From there we conclude that

$$z \in \bigcup_{i\in I} S(y_i).$$

Moreover, since $z \in conv\{y_i \, : i \, \in I\}$ is arbitrary, thus

$$conv\{y_i \, : i \, \in I\} \subset \bigcup_{i\in I} S(y_i).$$

According to Lemma of KKM [2] (see, Chapper 1, Theorem 24), we have

$$\cap_{i\in I} S(y_i) \neq \emptyset.$$

As $S(y)$ is closed subset for all $y \in D$ and moreover, D is compact set in X, $S(y) \subset D$, and this implies that

$$\cap_{y\in D} S(y) \neq \emptyset.$$

From there there exists at least one element $\bar{x} \in D$ such that

$$\bar{x} \in S(y), \text{ for all } y \in D.$$

Consequently, there exists at least one element $\bar{x} \in D$ such that

$$G(y,\bar{x}) - H(\bar{x},y) \not\subset intC, \text{ for all } y \in D.$$

For the last assertion, by taking into account Proposition 3.1.4, there exists a pointed closed convex cone $\widetilde{C}$ with $C \setminus \{0\} \subset int(\widetilde{C})$. In the same way as above, where $C := \widetilde{C}$, there exists a point $\bar{x} \in D$ such that

$$G(y,\bar{x}) - H(\bar{x},y) \not\subset int(\widetilde{C}).$$

Therefore there exists $\bar{x} \in D$ such that

$$G(y,\bar{x}) - H(\bar{x},y) \not\subset C \setminus \{0\}, \text{ for all } y \in D.$$

From there completing the proof.

Remark 3.1.5. The condition (E) in Theorem 3.1.3 is correct. Indeed, let $G,H,F : D \times D \longrightarrow 2^Y$ be three multivalued mappings with nonempty values, $G \neq H$ and F has the form $F(x,y) = G(y,x) - H(x,y)$ for all $(x,y) \in D \times D$, one can find some conditions for both G and H such that

F is upper C-continuous on $D \times D$. By direct applying Proposition 3.1.4 below, it follows that the set

$$A(y) := \{x \in D : G(y,x) - H(x,y) \subset intC\}$$

is open in D for each $y \in D$.

Proposition 3.1.6. *Let D be a nonempty subset in X and let C be a closed cone in Y. If the multivalued mapping $F : D \longrightarrow 2^Y$ with nonempty closed values is upper C-continuous on D, then the set $A := \{x \in D | F(x) \subset intC\}$ is open in D.*

Proof. If $intC = \emptyset$, nothing to prove. Conversely, let $\bar{x} \in A$ be arbitrary and $F(\bar{x}) \subset intC$. Since F is upper $C-$ continuous on D, hence F is upper $C-$ continuous at $\bar{x} \in D$. Then for any neighborhood W of the origin in Y, one can find a neighborhood U of $\bar{x}$ in $domF$ such that

$$F(x) \subset F(\bar{x}) + W + C, \text{ for all } x \in U.$$

Finally, we will check that for all $x \in U$ then $F(x) \subset intC$. Posit to the contrary that, there is $x_0 \in U$ with $F(x_0) \not\subset intC$. Then there is $y_0 \in F(x_0)$ but $y_0 \notin intC$. From there we infer that $y_0 \in F(\bar{x}) + W + C$. On the other hand, since W is an arbitrary neighborhood of the origin in Y, $F(\bar{x})$ is closed subset in Y and C is closed cone in Y, thus $y_0 \in F(\bar{x}) + C \subset intC + C = intC$ and it leads to a contradiction. Hence $U \cap D \subset A$ and the conclusion follows.

Proposition 3.1.7. *Let D and C be given as in Proposition 3.1.4. Then, if the multivalued mapping $F : D \longrightarrow 2^Y$ with nonempty closed values is lower C-continuous on D then the set $A := \{x \in D | F(x) \cap intC \neq \emptyset\}$ is open in D.*

Proof. If $A = \emptyset$, nothing to prove. Conversely, let $\bar{x} \in A$ be arbitrary such that $F(\bar{x}) \cap intC \neq \emptyset$. Since F is lower $C-$ continuous at $\bar{x}$, thus for any neighborhood W of the origin in Y one can find neighborhood U of $\bar{x}$ such that

$$F(\bar{x}) \subset F(x) + W - C, \text{ for all } x \in U.$$

We must show that for all $x \in U$ then $F(x) \cap intC \neq \emptyset$. In the converse case, there exists $x_0 \in U$ with $F(x_0) \cap intC = \emptyset$. Because $F(\bar{x}) \subset F(x_0) + W - C$, $F(x_0) - C$ is a closed subset in Y and W is an arbitrary neighborhood of the origin in Y hence $F(\bar{x}) \cap intC = \emptyset$ and we have a contradiction. So, the proposition 3.1.7 is proved complete.

Theorem 3.1.8. *Let G,H be as in Theorem 3.1.3. Suppose that all the following conditions are fulfilled*

(A) $G(x,x)=\{0\}$ and $H(x,x)=\{0\}$ for all $x\in D$;

(B) G is upper C-monotone*;

(C) For all $x\in D$, $x=\sum_{i\in I}\lambda_i x_i$, where $x_i\in D,\lambda_i\in[0;1]$ for all $i\in I$, $\sum_{i\in I}\lambda_i=1$, I is finite index set,

$$G(y,x)\subset\sum_{i\in I}\lambda_i G(y,x_i)-C, \text{ for all } y\in D.$$

(D) For all $x\in D$, the multivalued mapping $H(x,\ .\ ):D\longrightarrow 2^Y$ is C-convex;

(E) For all $y\in D$, the set $A(y):=\{x\in D: G(y,x)-H(x,y)\cap intC\neq\emptyset\}$ is open in D.

Then there exists a point $\overline{x}\in D$ such that

$$\Big(G(y,\overline{x})-H(\overline{x},y)\Big)\cap intC=\emptyset, \text{ for all } y\in D.$$

Moreover, assume that there exists a pointed closed convex cone $\widetilde{C}$ with $C\setminus\{0\}\subset int(\widetilde{C})$, then there exists a point $\overline{x}\in D$ such that

$$\Big(G(y,\overline{x})-H(\overline{x},y)\Big)\cap C\setminus\{0\}=\emptyset, \text{ for all } y\in D.$$

Proof. We first show that there exists a point $\overline{x}\in D$ such that

$$\Big(G(y,\overline{x})-H(\overline{x},y)\Big)\cap intC=\emptyset, \text{ for all } y\in D.$$

For any fixed $y\in D$, we put

$$S(y):=\{x\in D:\ \Big(G(y,x)-H(x,y)\Big)\cap intC=\emptyset\}.$$

Obviously, $S(y)$ is nonempty closed subset in D for all $y\in D$. We second prove that

$$\bigcap_{y\in D}S(y)\neq\emptyset.$$

In fact, let $\{y_i: i\in I\}$ be a finite arbitrary subset in D. Let us choose $z\in conv\{y_i: i\in I\}$ be arbitrary and then write z of the form

$$z=\sum_{i\in I}\lambda_i y_i, \text{ where } \lambda_i\in[0;1]\,\forall i\in I,\ \sum_{i\in I}\lambda_i=1.$$

From there we conclude that

$$z \in \underset{i \in I}{\cup} S(y_i).$$

In fact, it it were not so, then we get

$$z \notin \underset{i \in I}{\cup} S(y_i),$$

which yields that $z \notin S(y_i)$ for all $i \in I$. By the definition, we obtain as follows

$$\Big(G(y_i, z) - H(z, y_i)\Big) \cap int(C) \neq \emptyset, \text{ for all } i \in I.$$

Consequently

$$\sum_{i \in I} \lambda_i \Big(G(y_i, z) - H(z, y_i)\Big) \cap int(C) \neq \emptyset. \tag{5}$$

In view of the hypotheses of (B), the multivalued mapping G is upper $C-$ monotone and this means that

$$G(y_i, y_j) + G(y_j, y_i) \subset -C, \text{ for all } i, j \in I.$$

This combines with the hypotheses of (C), we obtain as follows

$$\begin{aligned} \sum_{i \in I} \lambda_i G(y_i, z) &= \sum_{i \in I} \lambda_i G(y_i, \sum_{j \in I} \lambda_j y_j) \subset \sum_{i,j \in I} \lambda_i \lambda_j G(y_i, y_j) - C \\ &= \frac{1}{2} \sum_{i,j \in I} \lambda_i \lambda_j \Big(G(y_i, y_j) + G(y_j, y_i)\Big) - C \\ &\subset - \sum_{i,j \in I} \lambda_i \lambda_j C - C \subset -C - C \subset -C. \end{aligned} \tag{6}$$

In other words, by (A) and (D), we also have

$$\begin{aligned} \sum_{i \in I} \lambda_i H(z, y_i) &\subset H(z, \sum_{i \in D} \lambda_i y_i) + C \\ &= H(z, z) + C = \{0\} + C = C. \end{aligned} \tag{7}$$

Combining (6)-(7), yields that

$$\sum_{i \in I} \lambda_i \Big(G(y_i, z) - H(z, y_i)\Big) \subset -C. \tag{8}$$

Since C is pointed cone, and this combines with both (5) and (8), we have a contradiction.

From there we conclude that

$$z \in \underset{i \in I}{\cup} S(y_i).$$

Moreover, since $z \in conv\{y_i\ : i\ \in I\}$ is arbitrary, thus

$$conv\{y_i\ : i\ \in I\} \subset \bigcup_{i\in I} S(y_i).$$

According to Lemma of KKM [2] (see, Chapper 1, Theorem 24), we have

$$\cap_{i\in I} S(y_i) \neq \emptyset.$$

Since $S(y)$ is closed subset for all $y \in D$ and moreover, D is compact set in X, $S(y) \subset D$, and this implies that

$$\cap_{y\in D} S(y) \neq \emptyset.$$

From there there exists a point $\overline{x} \in D$ such that

$$\overline{x} \in S(y),\ \text{for all } y \in D.$$

Consequently, there exists $\overline{x} \in D$ such that

$$\Big(G(y,\overline{x}) - H(\overline{x},y)\Big) \cap intC = \emptyset,\ \text{for all } y \in D.$$

For the last assertion, we assume that there exists a pointed closed convex cone $\widetilde{C}$ with $C\setminus\{0\} \subset int(\widetilde{C})$. By a similar argument as above, where $C := \widetilde{C}$, there exists a point $\overline{x} \in D$ such that

$$\Big(G(y,\overline{x}) - H(\overline{x},y)\Big) \cap int(\widetilde{C}) = \emptyset,\ \text{for all } y \in D.$$

So that there exists $\overline{x} \in D$ such that

$$\Big(G(y,\overline{x}) - H(\overline{x},y)\Big) \cap C\setminus\{0\} = \emptyset,\ \text{for all } y \in D.$$

From there completing the proof.

3.2. Some applications

In this subsection, we lead to some sufficient conditions on existence of solutions of general vector α optimization problems (where $\alpha \in \{P, I, W, Pr\}$) and the problems $(UPQEP)$, $(LPQUP)$, $(UWQEP)$ and $(LWQEP)$.

Theorem 3.2.1. *We suppose that all the following conditions are fulfilled*

(A) $G(y,x,x) = H(y,x,x) = \{0\}$ *for all* $y \in K,\ x \in D$;

(B) For all $y \in K$, the multivalued mapping $G(y, \, . \, , \, . \,) : D \times D \longrightarrow 2^Y$ is upper C-monotone*;

(C) For all $x \in D$, $x = \sum_{i \in I} \lambda_i x_i$, where $x_i \in D, \lambda_i \in [0;1]$ for all $i \in I$, $\sum_{i \in I} \lambda_i = 1$, I is finite index set,

$$G(y,z,x) \subset \sum_{i \in I} \lambda_i G(y,z,x_i) - C, \text{ for all } (y,z) \in K \times D;$$

(D) For all $(y,x) \in K \times D$, the multivalued mapping $H(y,x, \, . \,) : D \longrightarrow 2^Y$ is C-convex on D;

(E) For all $(y,x) \in K \times D$, the set $A(y,x) := \{x' \in D : G(y,x,x') - H(y,x',x) \subset intC\}$ is open in D;

(F) S has nonempty values and $D \setminus S(x) \subset M(x)$ for all $x \in D$, where $M : D \longrightarrow 2^D$ is given by

$$M(x) = \{x' \in D \,|\, G(y,x',x) - H(y,x,x') \subset intC, \text{ for some } y \in T(x,x')\}.$$

Then there exists a point $\bar{x} \in D$ such that $\bar{x} \in S(\bar{x})$ and

$$F(y,\bar{x},x) \not\subset -int(C), \text{ for all } x \in S(\bar{x}), \; y \in T(\bar{x},x).$$

Furthermore, assume that there exists a pointed closed convex cone $\widetilde{C}$ with $C \setminus \{0\} \subset int(\widetilde{C})$, then there exists a point $\bar{x} \in D$ such that $\bar{x} \in S(\bar{x})$ and

$$F(y,\bar{x},x) \not\subset -(C \setminus \{0\}), \text{ for all } x \in S(\bar{x}), \; y \in T(\bar{x},x).$$

Proof. For any fixed $y \in K$. According to Theorem 3.1.8, there exists $\bar{x} \in D$ such that

$$G(y,x,\bar{x}) - H(y,\bar{x},x) \not\subset intC \quad \text{for all } x \in D.$$

It is not difficult to see that $S(\bar{x}) \subset D$ and $T(\bar{x},x) \subset K$. Therefore

$$G(y,x,\bar{x}) - H(y,\bar{x},x) \not\subset intC \quad \text{for all } x \in S(\bar{x}), \; y \in T(\bar{x},x). \tag{9}$$

From there we must show that $\bar{x} \in S(\bar{x})$. Posit to the contrary that, $x \notin S(x)$ for all $x \in D$. We consider the multivalued mapping $M : D \longrightarrow 2^D$ is defined as

$$M(x) = \{x' \in D \,|\, G(y,x',x) - H(y,x,x') \subset intC, \text{ for some } y \in T(x,x')\}.$$

By hypotheses, for every $x \in D$, it follows that $S(x) \neq \emptyset$ and $D \setminus S(x) \subset M(x)$. Hence $x \in M(x)$ for all $x \in D$. Next, we consider the multivalued mapping $N : D \longrightarrow 2^D$ is defined as

$$N(x) = \{x' \in D | G(y,x',x) - H(y,x,x') \not\subset intC, \text{ for all } x \in S(x'), \; y \in T(x,x')\}.$$

It follows from (9) that $\overline{x} \in N(\overline{x})$ and $N(x) = D \setminus M(x)$ for all $x \in D$. Consequently, $x \notin N(x)$ for all $x \in D$ and this leads to a contradiction. So, there exists $\overline{x} \in D$ such that $\overline{x} \in S(\overline{x})$ and

$$F(y,\overline{x},x) \not\subset -int(C), \text{ for all } x \in S(\overline{x}),\ y \in T(\overline{x},x).$$

On the other hand, by hypotheses there exists a pointed closed convex cone $\widetilde{C}$ with $C \setminus \{0\} \subset int(\widetilde{C})$. By a similar argument as above, where $C := \widetilde{C}$, there exists $\overline{x} \in D$ such that $\overline{x} \in S(\overline{x})$ and

$$F(y,\overline{x},x) \not\subset -(C \setminus \{0\}), \text{ for all } x \in S(\overline{x}),\ y \in T(\overline{x},x),$$

which the claim follows.

Theorem 3.2.2. *Assume that all the following conditions are fulfilled*

(A) $G(y,x,x) = H(y,x,x) = \{0\}$ *for all* $y \in K,\ x \in D$;

(B) For all $y \in K$, *the multivalued mapping* $G(y, . , .) : D \times D \longrightarrow 2^Y$ *is upper C-monotone*;*

(C) For all $x \in D,\ x = \sum_{i\in I} \lambda_i x_i$, *where* $x_i \in D, \lambda_i \in [0;1]$ *for all* $i \in I$, $\sum_{i\in I} \lambda_i = 1$, *I is finite index set,*

$$G(y,z,x) \subset \sum_{i\in I} \lambda_i G(y,z,x_i) - C, \text{ for all } (y,z) \in K \times D.$$

(D) For all $(y,x) \in K \times D$, *the multivalued mapping* $H(y,x, .) : D \longrightarrow 2^Y$ *is C-convex;*

(E) For all $(y,x) \in K \times D$, *the set* $A(y,x) := \{x' \in D : \Big(G(y,x,x') - H(y,x',x)\Big) \cap intC \neq \emptyset\}$ *is open in D;*

(F) S has nonempty values and $D \setminus S(x) \subset M(x)$ *for all* $x \in D$, *where* $M : D \longrightarrow 2^D$ *is given by*

$$M(x) = \{x' \in D \,|\, \Big(G(y,x',x) - H(y,x,x')\Big) \cap intC \neq \emptyset, \text{ for some } y \in T(x,x')\}.$$

Then there exists a point $\overline{x} \in D$ *such that* $\overline{x} \in S(\overline{x})$ *and*

$$F(y,\overline{x},x) \cap -int(C) = \emptyset, \text{ for all } x \in S(\overline{x}),\ y \in T(\overline{x},x).$$

Furthermore, assume that there exists a pointed closed convex cone $\widetilde{C}$ *with* $C \setminus \{0\} \subset int(\widetilde{C})$, *then there exists* $\overline{x} \in D$ *such that* $\overline{x} \in S(\overline{x})$ *and*

$$F(y,\overline{x},x) \cap -(C \setminus \{0\}) = \emptyset, \text{ for all } x \in S(\overline{x}),\ y \in T(\overline{x},x).$$

Proof. For any fixed $y \in K$. Making use of Theorem 3.1.8 we get there exists $\overline{x} \in D$ such that

$$\Big(G(y,x,\overline{x}) - H(y,\overline{x},x)\Big) \cap intC = \emptyset \quad \text{for all } x \in D.$$

It is clear that $S(\overline{x}) \subset D$ and $T(\overline{x},x) \subset K$. Thus

$$\Big(G(y,x,\overline{x}) - H(y,\overline{x},x)\Big) \cap intC = \emptyset \quad \text{for all } x \in S(\overline{x}),\ y \in T(\overline{x},x). \tag{10}$$

To finish the proof we must prove that $\overline{x} \in S(\overline{x})$. Posit to the contrary that, $x \notin S(x)$ for all $x \in D$. We define the multivalued mapping $M : D \longrightarrow 2^D$ is given by

$$M(x) = \{x' \in D \,|\, \Big(G(y,x',x) - H(y,x,x')\Big) \cap intC \neq \emptyset, \text{ for some } y \in T(x,x')\}.$$

By hypotheses, for all $x \in D$, $S(x) \neq \emptyset$ and $D \setminus S(x) \subset M(x)$. Hence $x \in M(x)$ for all $x \in D$. We next consider the multivalued mapping $N : D \longrightarrow 2^D$ by

$$N(x) = \{x' \in D \,|\, \Big(G(y,x',x) - H(y,x,x')\Big) \cap intC = \emptyset, \text{ for all} x \in S(x'), y \in T(x,x')\}.$$

By direct applying (10), yields that $\overline{x} \in N(\overline{x})$ and $N(x) = D \setminus M(x)$ for all $x \in D$. Consequently $x \notin N(x)$ for all $x \in D$ and this is a contradiction. So, there exists $\overline{x} \in D$ such that $\overline{x} \in S(\overline{x})$ and

$$F(y,\overline{x},x) \cap -int(C) = \emptyset, \text{ for all } x \in S(\overline{x}),\ y \in T(\overline{x},x).$$

Suppose, in addition, that there exists a pointed closed convex cone $\widetilde{C}$ with $C \setminus \{0\} \subset int(\widetilde{C})$. In the similar way as above, where $C := \widetilde{C}$, there exists $\overline{x} \in D$ such that $\overline{x} \in S(\overline{x})$ and

$$F(y,\overline{x},x) \cap -(C \setminus \{0\}) = \emptyset, \text{ for all } x \in S(\overline{x}),\ y \in T(\overline{x},x)$$

and the claim follows.

Theorem 3.2.3. *Assume that the $C-$ convex vector valued function $F : D \longrightarrow Y$ is upper C-continuous on D. Then*

(A) There exists $\overline{x} \in D$ such that

$$(GVOP)_W : \quad F(\overline{x}) \cap WMin\Big(F(D)|C\Big) \neq \emptyset.$$

Furthermore, if there exists a pointed closed convex cone $\widetilde{C}$ with $C\setminus\{0\}\subset int(\widetilde{C})$, then there exists $\overline{x}\in D$ such that

$$(GVOP)_P: \quad F(\overline{x})\cap PMin\Big(F(D)|C\Big)\neq\emptyset.$$

In addition we also have

$$PMin(F(D)\,|\,C)\neq\emptyset, \qquad WMin(F(D)\,|\,C)\neq\emptyset.$$

(B) If the problem $(GVOP)_{Pr}$ has solutions then the problems $(GVOP)_\alpha$ for $\alpha\in\{P,I,W\}$ also has solutions.

Proof. Let us consider the multivalued mappings H and G from $D\times D$ into 2^Y be given respectively as $H(x,y)=\{F(y)-F(x)\},\ \forall(x,y)\in D\times D$ and $G(x,y)=\{0\},\ \forall(x,y)\in D\times D$. Obviously, G satisfies all the conditions (A), (B) and (C) of Theorem 3.1.3 and furthermore $H(x,x)=\{F(x)-F(x)\}=\{0\}$ for all $x\in D$. So that condition (A) is satisfied. For any fixed $x\in D$, we must show that the multivalued mapping $H(x,.):D\longrightarrow 2^Y$ is C-convex. In fact, for any $a,b\in D$ and $t\in[0;1]$, if we pick $z=ta+(1-t)b$ then $z\in D$ as D is convex subset in X. By hypotheses, for any $x\in D$, it is obvious that $tF(x)+(1-t)F(x)=F(x)$. Consequently

$$\begin{aligned}tH(x,a)+(1-t)H(x,b)&=tF(a)+(1-t)F(b)-\Big(tF(x)+(1-t)F(x)\Big)\\&\subset F(ta+(1-tb))+C-\Big(tF(x)+(1-t)F(x)\Big)\\&\subset F(z)-F(x)+C=H(x,z)+C,\end{aligned}$$

yields that the multivalued mapping $H(x,.)$ is C-convex. Finally, for all $y\in D$, in view of Remark 3.1.2, the set $A(y):=\{x\in D:\ F(x)-F(y)\subset intC\}$ is open on D. By taking account of Theorem 3.1.3 there exists $\overline{x}\in D$ such that $F(\overline{x})-F(y)\not\subset int(C)$, for all $y\in D$. From here we conclude that there exists $\overline{y}\in F(\overline{x})$ such that

$$F(D)\cap\Big(\{\overline{y}\}-int(C)\Big)=\emptyset.$$

According to Luc [7, Proposition 2.3, p. 41-42], yields that

$$\overline{y}\in F(\overline{x})\cap WMin(F(D)\,|\,C).$$

Therefore the problem $(GVOP)_W$ has solution. Furthermore, there exists $\bar{x} \in D$ such that

$$F(\bar{x}) - F(y) \not\subset (C \setminus \{0\}), \text{ for all } y \in D,$$

then there exists $\bar{y} \in F(\bar{x})$ such that $F(D) \cap (\bar{y} - C) = \{\bar{y}\}$. Making use of Proposition 2.3 in Luc [7, p. 41-42], we get

$$\bar{y} \in F(\bar{x}) \cap PMin(F(D) \,|\, C).$$

It means that the problem $(GVOP)_P$ has solution. For the last assertion: Obviously, if the problem $(GVOP)_{Pr}$ has solutions, then by using Proposition 2.3 of Luc [7], we get the problems $(GVOP)_P$, $(GVOP)_I$ and $(GVOP)_W$ has also solutions. The proof is completed.

As applications of the theorems 3.1.3 and 3.1.8, a sufficient condition on the existence of equilibrium points of the vector valued function $T : D_1 \times D_2 \subset X_1 \times X_2 \longrightarrow Y$ with respect to C is stated as follows

Theorem 3.2.4. *Let X_1, X_2 and Y be Hausdorff locally convex topological vector spaces, let $D_1 \subset X_1,\ D_2 \subset X_2$ be nonempty compact convex subsets in X_1 and X_2 respectively and let C be a pointed closed convex cone in Y. For a given vector valued function $T : D_1 \times D_2 \longrightarrow Y$. Assume, in addition, that the following conditions are fulfilled*

(A) T is C-convex and upper C-continuous in the first variable;

(B) T is C-concave and lower C-continuous in the second variable;
Then there exists the pair $(\overline{x_1}, \overline{x_2}) \in D_1 \times D_2$ such that

$$T(y_1, \overline{x_2}) - T(\overline{x_1}, y_2) \not\subset -intC \quad \textit{for all } (y_1, y_2) \in D_1 \times D_2.$$

In addition, we assume that there exists a pointed closed convex cone $\widetilde{C}$ with $C \setminus \{0\} \subset int(\widetilde{C})$, then there exists a point $(\overline{x_1}, \overline{x_2}) \in D_1 \times D_2$ such that

$$T(y_1, \overline{x_2}) - T(\overline{x_1}, y_2) \not\subset -(C \setminus l(C)) \quad \textit{for all } (y_1, y_2) \in D_1 \times D_2.$$

Proof. Firstly, we prove that there exists the pair $(\overline{x_1}, \overline{x_2}) \in D_1 \times D_2$ such that

$$T(y_1, \overline{x_2}) - T(\overline{x_1}, y_2) \not\subset -intC \quad \text{for all } (y_1, y_2) \in D_1 \times D_2. \tag{11}$$

We set

$$X = X_1 \times X_2, \quad D = D_1 \times D_2.$$

Consider the multivalued mappings G and H from $D \times D$ into 2^Y are defined respectively by

$$H(x,y) = \{T(y_1,x_2) - T(x_1,y_2)\} \quad \text{for all } x = (x_1,x_2), y = (y_1,y_2) \in D$$

and $$G(x,y) = \{0\} \quad \text{for all } x = (x_1,x_2), y = (y_1,y_2) \in D.$$

It is clear that the multivalued mapping G satisfies all the conditions (A), (B) and (C) of Theorem 3.1.3 and the multivalued mapping H satisfies the condition (A). For any fixed $x = (x_1,x_2) \in D$, we show that $H(x,.) : D \longrightarrow 2^Y$ is C-convex. In fact, for all $a = (a_1,a_2),\ b = (b_1,b_2) \in D$, for all $t \in [0;1]$, denote by $z_i = ta_i + (1-t)b_i \in D_i,\ i = 1,2$. Since D_1, D_2 are nonempty convex subsets in X_1 and X_2 respectively, thus $D = D_1 \times D_2$ is a nonempty convex subset in $X = X_1 \times X_2$. Let us fix $z = (z_1,z_2) \in D$. Since T is C-convex in the first variable and C-concave in the second variable and $C + C \subset C$, hence the following inclusions hold

$$\begin{aligned} tH(x,a) + (1-t)H(x,b) &= \{tT(a_1,x_2) + (1-t)T(b_1,x_2)\} - \{tT(x_1,a_2) + (1-t)T(x_1,b_2)\} \\ &\subset \{T(ta_1 + (1-tb_1,x_2))\} + C - \{T(x_1,ta_2 + (1-t)b_2)\} \\ &\subset \{T(z_1,x_2) - T(x_1,z_2)\} + C + C \subset H(x,z) + C. \end{aligned}$$

Thus the multivalued mapping $H(x,.)$ is C-convex on D. Finally for all $y = (y_1,y_2) \in D$, we show that the set

$$\begin{aligned} A(y) &:= \{x = (x_1,x_2) \in D = D_1 \times D_2 : G(y,x) - H(x,y) \subset intC\} \\ &= \{x \in D \,|\, \{T(x_1,y_2)\} - \{T(y_1,x_2)\} \subset intC\} \end{aligned}$$

is open on D. Because $T(.,y_2)$ is upper C-continuous in the first variable and $T(y_1,.)$ is lower C-continuous in the second variable, thus $T(.,y_2) - T(y_1,.)$ is upper C-continuous. Making use of Remark 3.1.2, we get $A(y)$ is open subset in D. Furthermore, by using Theorem 3.1.2, yields there exists $\bar{x} = (\overline{x_1},\overline{x_2}) \in D = D_1 \times D_2$ such that

$$G(y,\bar{x}) - H(\bar{x},y) \not\subset int(C), \text{ for all } y = (y_1,y_2) \in D = D_1 \times D_2.$$

This leads to there exists $(\overline{x_1},\overline{x_2}) \in D$ such that (11) holds. Finally, we suppose that there exists a pointed closed convex cone $\widetilde{C}$ with $C \setminus \{0\} \subset int(\widetilde{C})$, then by virtue of Theorem 3.1.2, where

$C := \widetilde{C}$, there exists $\bar{x} = (\overline{x_1}, \overline{x_2}) \in D = D_1 \times D_2$ such that

$$G(y, \bar{x}) - H(\bar{x}, y) \not\subset -(C \setminus l(C)), \text{ for all } y = (y_1, y_2) \in D = D_1 \times D_2.$$

This implies there exists $(\overline{x_1}, \overline{x_2}) \in D_1 \times D_2$ such that $T(y_1, \overline{x_2}) - T(\overline{x_1}, y_2) \not\subset -(C \setminus l(C))$ for all $(y_1, y_2) \in D_1 \times D_2$, which completes the proof.

Theorem 3.2.5. *Let X_i ($i \in I$, $card(I) = n$) be Hausdorff locally convex topological vector spaces. For each $i \in I$, let $D_i \subset X_i$ be nonempty compact subsets. Let*

$$D = D_1 \times D_2 \times \ldots \times D_n = \prod_{i=1}^{n} D_i.$$

For every $i \in I$, $F_i : D \longrightarrow Y$. Assume, furthermore, that for each $i \in I$ such that

(i) $F_i : D \longrightarrow Y$ is $C-$ continuous on D;

(ii) $x^i = \{x_j\}_{j \in I \setminus i} \in D \setminus D_i$;

(iii) The vector valued functions $F_i(x^i, \,.\,) : D_i \longrightarrow Y$ are $C-$convex on D_i.
Then there exists $\bar{x} = (\overline{x_i})_{i \in I} \in D$ such that for all $i \in I$,

$$F_i(\overline{x^i}, y_i) - F_i(\bar{x}) \notin -intC, \textit{ for all } (y_i)_i \in D.$$

In addition, we assume that there exists a pointed closed convex cone $\widetilde{C}$ such that $C \setminus \{0\} \subset int(\widetilde{C})$, then there exists $\bar{x} = (\overline{x_i})_{i \in I} \in D$ such that for all $i \in I$,

$$F_i(\overline{x^i}, y_i) - F_i(\bar{x}) \notin -(C \setminus l(C)), \textit{ for all } (y_i)_i \in D.$$

Proof. Let $X = \prod_{i=1}^{n} X_i$ and consider the multivalued mappings $G, H : D \times D \longrightarrow 2^Y$ be given respectively as

$$G(x, y) = \{0\},$$
$$H(x, y) = \{\sum_{i=1}^{n} (F_i(x^i, y_i) - F_i(x))\}$$

for all $x = (x_i)_{i \in I}, y = (y_i)_{i \in I} \in D$. By a similar argument as in the proof of Theorem 3.2.5, yields there exists $\bar{x} = (\overline{x_i})_{i \in I} \in D$ such that

$$G(y, \bar{x}) - H(\bar{x}, y) \not\subset intC, \text{ for all } y = (y_i)_{i \in I} \in D.$$

In other words,

$$G(y,\overline{x})-H(\overline{x},y)=-\{\sum_{i=1}^{n}(F_i(x^i,y_i)-F_i(x))\}\not\subset intC,$$

for all $y=(y_i)_{i\in I}\in D$. Now, for any $i\in I$, $x_i\in D_i$ be arbitrary, let $y=(\overline{x_i},y_i)$, then we obtain

$$-\Big(F_i(\overline{x}^i,y_i)-F_i(\overline{x})\Big)\notin intC,$$

which is equivalent to

$$F_i(\overline{x}^i,y_i)-F_i(\overline{x})\notin -intC.$$

From there we conclude that there is a point $\overline{x}$ such that for all $i\in I$,

$$F_i(\overline{x^i},y_i)-F_i(\overline{x})\notin -intC,\text{ for all }(y_i)_i\in D.$$

Moreover, we assume that there exists a pointed closed convex cone $\widetilde{C}$ such that $C\setminus\{0\}\subset int(\widetilde{C})$, then in a similar way as above, it follows that there exists $\overline{x}\in D$ such that

$$F_i(\overline{x^i},y_i)-F_i(\overline{x})\notin -(C\setminus l(C)),\text{ for all }(y_i)_i\in D,\ \forall\, i\in I,$$

which completes the proof.

Acknowledgement

The authors are grateful to the reviewers for useful suggestions which improve the contents of this paper.

REFERENCES

[1] Q.H. Ansari, Vector Equilibrium Problems and Vector Variational Inequalites, Presented at 2nd World Congress of Nonlinear Analysis, Athens, (1996).

[2] J.P. Aubin, I. Ekland, Applied Nonlinear Analysis, Wiley Inter-Science, (1984).

[3] E. Blum, W. Oettli, From optimization and variational inequalites to equilibrium problems, Math. Stud. 63 (1994), 123-145.

[4] T.V. Su, Fritz John type optimality conditions for weak efficient solution of vector equilibrium problems with constraints in terms of contingent epiderivatives, Appl. Math. Sci. 9 (2015), 6249-6261.

[5] D. Chang, J.S. Pang, The generalized quasi-variational inequality problem, Math. Oper. Res. 7 (1982), 211-222.

[6] A. Gurraggio, X.N. Tan, On general vector quasi-optimization problems, Math. Meth. Oper Res. 55 (2002), 347-358.

[7] D.T. Luc, Theory of vector optimization, Lect. notes in Eco. and Math. systems, Springer Verlag, Berlin, Germany, Vol 319, 1989.

[8] D.T. Luc, N.X. Tan, Existance conditions in variational inclusion with constraint, Optim. 53 (2004), 505-515.

[9] D.T. Luc, N.X. Tan, P.N. Tinh, Convex vector functions and their subdifferential, Acta. Math. Viet. 28 (1998), 107-127.

[10] L.J. Lin, Z.T. Yu, G. Kassay, Existance of equilibria for monotone multivaluted mappings and its applications to vectorial equilibria, J. Optim. Theory Appl. 114 (2002), 189-208.

[11] N.B. Minh, N.X. Tan, Some sufficient conditions for the existence of equilibrium points concerning multivaluted mappings, Vietnam J. Math. 28 (2000), 259-310.

[12] N.B. Minh, N.X. Tan, On the existence of solutions of quasivariational inclusion problems of Stampachia type, Adv. Nonlinear Var. Inequal. 8 (2005), 1-16.

[13] J.J. Moreau, Functional Analysis and Optimization, Academic Press, New York, 1966.

[14] S. Park, Fixed points and quasi-equilibrium problems, Nonlinear Oper. Theory Math. Comput. Model. 32 (2000), 1297-1304.

[15] J. Parida, A. Sen, A Variational like inequality for multifuntions with applications, J. Math. Anal. Appl. 124 (1987), 73-81.

[16] Y. Sawaragi, H. Nakayama, T.Tanino, Theory of multiobjective optimization, Academic Press INC., New York and London (1985).

[17] N.X. Tan, On the existence of solutions of quasivariational inclusion problems, J. Optim. Theory Appl. 123 (2004), 619-638.

[18] L.J. Lin, N.X. Tan, On Systems of quasivariational inclusion problems of type I and related problems, Vietnam J. Math. 34 (2006), 423-440.

[19] N.C. Yannelis, N. D. Prabhaker, Existance of minimax elements and equilibria in linear topological spaces, J. Math. Eco. 12 (1983), 233-245.

[20] T.V. Su, Second-order optimality conditions for vector equilibrium problems, J. Nonlinear Funct. Anal. 2015 (2015), Article ID 6.

[21] M. Bianchi, N. Hadjisavvas, S. Schaible, Vector equilibrium problems with generalized monotone bifunctions, J. Optim. Theory Appl. 92 (1997), 527-542.