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CONVERGENCE OF A BREGMAN PROJECTION ALGORITHM FOR MONOTONE OPERATORS IN BANACH SPACES

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Abstract. The purpose of this paper is to investigate the convergence analysis of a Bregman projection algorithm. Strong convergence theorems for zeros of monotone operators and solutions to a generalized equilibrium problems is established in a reflexive Banach space.

Keywords. Bregman projection; Convergence; Monotone operator; Zero point.

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1. Introduction

Equilibrium problems provide us with a simple, general and unified framework to study a wide class of problems arising in pure and applied sciences. During the last two decades, there has been considerable activity in the development of numerical techniques for solving equilibrium problems; see [1]-[6] and the references therein. Mean valued iterative algorithm is powerful when dealing with the equilibrium problems. Krasnoselski-Mann iterative algorithm (KMIA) generates a sequence $\{x_n\}$ in the following manner: $x_0 \in C$, $x_{n+1} = \alpha_n T x_n + (1-\alpha_n)x_n$, $\forall n \geq 0$, It is known that (KMIA) only has weak convergence even for equilibrium problems in infinite-dimensional Hilbert spaces. In many disciplines, problems arises in infinite dimension spaces. Strong convergence is often much more desirable than weak convergence. To improve the weak convergence of the Krasnoselski-Mann iterative algorithm, so called hybrid projections have been considered; see [7]-[11] for more details and the references therein.

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The purposes of this paper is to study (KMIA) with the help of additional Bregman projections. The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a new bregman projection algorithm is studied. Strong convergence theorems are established in Banach spaces.

2. Preliminaries

Let E be a real Banach space. Let E^* be the dual space of E. Let C be a nonempty subset of a E. For any convex function $g : E \to (-\infty, +\infty]$, we denote the domain of g by $Dom(g) = \{x \in E : g(x) < \infty\}$. For any $x \in intDom(g)$ and $y \in E$, we denote by $g^0(x,y)$ the right-hand derivative of g at x in the direction y, that is, $g^0(x,y) = \lim_{t\to 0^+} \frac{g(x+ty)-g(x)}{t}$. The function g is said to be Gâteaux differentiable at x if $\lim_{t\to 0^+} \frac{g(x+ty)-g(x)}{t}$ exists for any y. In this case, $g^0(x,y)$ coincides with $\nabla g(x)$, the value of the gradient ∇g of g at x. The function g is said to be Gâteaux differentiable if it is Gâteaux differentiable everywhere. The function g is said to be Fréchet differentiable at x if the limit is attained uniformly in $\|y\| = 1$. The function g is said to be Fréchet differentiable if it is Fréchet differentiable everywhere. It is well known that if a continuous convex function $g : E \to \mathbb{R}$ is Gâteaux differentiable, then ∇g is norm-to-weak* continuous. If $g : E \to \mathbb{R}$ is Fréchet differentiable, then ∇g is norm-to-norm continuous; for more details. The function g is said to be strongly coercive [12] if $\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = \infty$. It is said to be bounded on bounded subsets of E if $g(B)$ is bounded for each bounded subset B of E. Let E be a reflexive Banach space. For any proper, lower semicontinuous and convex function: $g : E \to (-\infty, +\infty]$, the conjugate function g^* of g is defined by $g^*(x^*) = \sup_{x\in E}\{\langle x, x^*\rangle - g(x)\}, \forall x^* \in E^*$. It is well known that $g(x) + g^*(x^*) \geq \langle x, x^*\rangle$ for all $(x, x^*) \in E \times E^*$. It is also known that $(x, x^*) \in \partial g$, where ∂g is the subdifferential of g, is equivalent to $g(x) + g^*(x^*) = \langle x, x^*\rangle$. If g is a proper, lower semicontinuous and convex function, then g^* is a proper, weak* lower semicontinuous and convex function. Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then Bregman distance corresponding to g is the function $D_g : E \times E \to \mathbb{R}$ defined by $D_g(x,y) = g(x) - g(y) - \langle x - y, \nabla g(y)\rangle, \forall x, y \in E$. It is clear that $D_g(x,y) \geq 0$ for all $x, y \in E$. If E is smooth and $g(x) = \|x\|^2$ for all $x \in E$, we obtain that $\nabla g(x) = 2Jx$, where J is the generalized duality mapping. If C is a nonempty, closed and convex subset of a reflexive Banach space E and

g is a strongly coercive Bregman function, then for each $x \in E$, there exists a unique $x_0 \in C$ such that $D_g(x_0,x) = \min_{y\in C} D_g(y,x)$. Bregman projection $Proj_C^g$ from E onto C is defined by $Proj_C^g x = x_0$ for all $x \in E$. It is also well known [12] that $Proj_C^g$ has the following property: $D_g(y, Proj_C^g x) + D_g(Proj_C^g x, x) \leq D_g(y,x), \forall y \in C, x \in E$. Let $B_r := \{z \in E : \|z\| < r\}$. A function $g : E \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of E if $f_r(t) > 0$ for all $r,t > 0$, where $f_r : [0,\infty] \to \infty[0,\infty]$ is defined by:

$$f_r(t) = \inf_{x,y\in B_r, \|x-y\|=t, \alpha\in(0,1)} \frac{\alpha g(x) + (1-\alpha)g(y) - g(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)}, \quad \forall t \geq 0.$$

The function f_r is called the gauge of the uniform convexity of g. If $g : E \to \mathbb{R}$ is a convex function which is uniformly convex on bounded subsets. Then $g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y) - \alpha(1-\alpha) f_r(\|x-y\|)$ for all $x, y \in B_r$ and $\alpha \in (0,1)$, where f_r is the gauge of the uniform convexity of g. Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers.

Let $B : C \to E^*$ be continuous and monotone mapping and let $\varphi : C \to R$ be a lower semi-continuous and convex function. Recall that the following generalized equilibrium problem. Find $\bar{x} \in C$ such that

$$f(\bar{x}, y) + \langle B\bar{x}, y - \bar{x}\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{2.1}$$

We use $EP(f,B,\varphi)$ to denote the solution set of the equilibrium problem (2.1). If $B = 0$ and $\varphi = 0$, then we have the following equilibrium problem: $f(\bar{x} y) \geq 0, \quad \forall y \in C$.

Let A be a multivalued operator from E to E^* with domain $Dom(A) = \{z \in E : Az \neq \emptyset\}$ and range $Ran(A) = \cup\{Az : z \in Dom(A)\}$. An operator A is said to be monotone iff $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ for each $x_i \in Dom(A)$ and $y_i \in Ax_i$, $i = 1,2$. A monotone operator A is said to be maximal if its graph $Grap(A) = \{(x,y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. We know that if A is a maximal monotone operator, then $A^{-1}(0)$ is closed and convex.

Let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the fixed point set of T. T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$.

Let $g : E \to \mathbb{R}$ is a proper, lower semicontinuous and convex function. Recall that $T : C \to C$ is said to be Bregman quasi-nonexpansive iff $F(T) \neq \emptyset$, $D_g(p,Tx) \leq D_g(p,x)$, $\forall x \in C, p \in F(T)$. A point p in C is said to be an asymptotic fixed point of T iff C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. T is said to be Bregman relatively nonexpansive iff $\widetilde{F}(T) = F(T) \neq \emptyset$ and $D_g(p,Tx) \leq D_g(p,x)$ for all $x \in C$ and $p \in F(T)$.

Let E be a reflexive, strictly convex and smooth Banach space, and let A be a maximal monotone operator from E to E^*. From Rockafellar [13], we find that $s > 0$ and $x \in E$, there exists a unique $x_s \in D(A)$ such that $\nabla gx \in \nabla gx_s + sAx_s$. If $J_s x = x_s$, then we can define a single valued mapping $J_s : E \to Dom(A)$ by $J_s = (\nabla g + sA)^{-1}\nabla g$ and such a J_s is called the resolvent of A. We know that J_s is closed and $A^{-1}(0) = F(J_s)$ for all $s > 0$. From [14], we know that $J_s : E \to Dom(A)$ is a Bregman quasi-nonexpansive.

In order to investigate equilibrium problem (2.1), we assume that f satisfies the following conditions:

(A1) $f(x,x) = 0, \forall x \in C$;

(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0, \forall x,y \in C$;

(A3) $\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x,y), \forall x,y,z \in C$;

(A4) for each $x \in C$, $y \mapsto f(x,y)$ is convex and weakly lower semi-continuous.

Lemma 2.1. [14] *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of E. Let $T : C \to C$ be a Bregman relatively nonexpansive mapping. Then $F(T)$ is closed and convex.*

Lemma 2.2. [15] *Let E be a Banach space and let $g : E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E. Then* $\lim_{n\to\infty} D_g(x_n,y_n) = 0 \Longleftrightarrow \lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.3. [16] *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous and strongly coercive function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of E and let f be* *a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$. Let $r > 0$ and $x \in E$. Then There exists $z \in C$ such that $rf(z,y) + \langle y - z, \nabla gz - \nabla gx \rangle \geq 0, \quad \forall y \in C$. Define a mapping $S_r : E \to C$ by $S_r x = \{z \in C : rf(z,y) + \langle y - z, \nabla gz - \nabla x \rangle, \quad \forall y \in C\}$. Then the following conclusions hold:*

(1) *S_r is single-valued Bregman quasi-nonexpansive.*

(2) $\langle S_r x - S_r y, \nabla g S_r x - \nabla g S_r y \rangle \leq \langle S_r x - S_r y, \nabla x - \nabla g y \rangle, x, y \in E$.

(3) *$F(S_r) = EP(f)$ is closed and convex.*

(4) $D_g(q, S_r x) + D_g(S_r x, x) \leq \phi(q, x)$, $\forall q \in F(S_r)$.

3. Main results

Theorem 3.1. *Let E be a reflexive Banach space and let $g : E \to R$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of E. Let C be a nonempty, closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$ and let $A : E \to E^*$ be a maximal monotone operator such that $Dom(A) \subset C$. Let $B : C \to E^*$ be continuous and monotone mapping and let $\varphi : C \to R$ be a lower semi-continuous and convex function. Assume that the common solution set $A^{-1}(0) \cap EF(f, B, \varphi)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$
\begin{cases}
x_0 \in E \quad \text{chosen arbitrarily}, \\
C_1 = C, \\
x_1 = Proj^g_{C_1} x_0, \\
y_n = \nabla g^*(\alpha_n \nabla g(x_n) + (1 - \alpha_n) \nabla g J^A_s x_n), \\
u_n \in C \text{ such that } r_n f(u_n, y) + r_n(\varphi(y) - \varphi(u_n)) + r_n \langle Bu_n, y - u_n \rangle + \langle y - u_n, \nabla g(u_n) - \nabla g(y_n) \rangle \geq 0, \\
C_{n+1} = \{z \in C_n : D_g(z, u_n) \leq D_g(z, x_n)\}, \\
x_{n+1} = Proj^g_{C_{n+1}} x_1,
\end{cases}
$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, ∇g is the right-hand derivative of g, $J^A_s = (\nabla g + sA)^{-1} \nabla g$, and $\{r_n\}$ is a real sequence such that $\liminf_{n\to\infty} r_n > 0$.

Then the sequence $\{x_n\}$ converges strongly to $Proj^g_{A^{-1}(0)\cap EF(f,B,\varphi)}x_1$, where $Proj^g_{A^{-1}(0)\cap EF(f,B,\varphi)}$ is the Bregman projection from E onto $A^{-1}(0)\cap EF(f,B,\varphi)$.

Proof. Put $F(x,y) = f(x,y) + \langle Bx, y-x\rangle + \varphi(y) - \varphi(x)$, $\forall x,y \in C$. Next, we prove that bifunction F satisfies (A1), (A2), (A3) and (A4). Therefore, the generalized equilibrium problem is equivalent to the following equilibrium problem: find $x \in C$ such that $F(x,y) \geq 0$, $\forall y \in C$. First, we prove F is monotone. Since B is a continuous and monotone operator, we find from the definition of F that $F(b,c) + F(c,b) \leq 0$. It is clear that F satisfies (A2). Next, we show that for each $a \in C$, $b \longmapsto F(a,b)$ is a convex and lower semicontinuous. For each $a \in C$, for all $t \in (0,1)$ and for all $b,c \in C$, since φ is convex, we have

$$\begin{aligned}
&F(a,tb+(1-t)c)\\
&= f(a,tb+(1-t)c) + \langle Ba, tb+(1-t)c-a\rangle + \varphi((tb+(1-t)c)) - \varphi(a)\\
&\leq t\big(f(a,b)+\varphi(b)-\varphi(a)+\langle Ba,b-a\rangle\big)\\
&\quad + (1-t)\big(f(a,c)+\varphi(c)-\varphi(a)+\langle Ba,c-a\rangle\big)\\
&= (1-t)F(a,c) + tF(a,b).
\end{aligned}$$

So, $b \longmapsto F(a,b)$ is convex. Similarly, we find that $b \longmapsto F(a,b)$ is also lower semicontinuous. Since B is continuous and φ is lower semicontinuous, we have

$$\begin{aligned}
\limsup_{t\downarrow 0} F(tc+(1-t)a,b) &= \limsup_{t\downarrow 0} f(tc+(1-t)a,b)\\
&\quad + \limsup_{t\downarrow 0}\big(\varphi(b) - \varphi((tc+(1-t)a))\big)\\
&\quad + \limsup_{t\downarrow 0}\langle B\big(tc+(1-t)a\big), b-\big(tc+(1-t)a\big)\rangle\\
&\leq f(a,b) + \varphi(b) - \varphi(a) + \langle Ba, b-a\rangle\\
&= F(a,b).
\end{aligned}$$

Hence $EF(f,B,\varphi)$ is closed and convex.

Next, we prove that C_n is closed, and convex. This can be proved by induction on n. It is obvious that $C_1 = C$ is closed and convex. Assume that C_m is closed and convex for some $m \geq 1$. For $z_1, z_2 \in C_{m+1}$, we see that $z_1, z_2 \in C_m$. It follows that $z = tz_1 + (1-t)z_2 \in C_m$, where $t \in (0,1)$. Notice that $D_g(z_1,u_m) \le D_g(z_1,x_m)$, and $D_g(z_2,u_m) \le D_g(z_2,x_m)$, The above inequalities are equivalent to

$$\langle z_1, \nabla g(x) - \nabla g(u_m)\rangle \le g(u_m) - g(x_m) - \langle u_m, \nabla g(u_m)\rangle + \langle x, \nabla g(x)\rangle, \tag{3.1}$$

and

$$\langle z_2, \nabla g(x) - \nabla g(u_m)\rangle \le g(u_m) - g(x_m) - \langle u_m, \nabla g(u_m)\rangle + \langle x, \nabla g(x)\rangle. \tag{3.2}$$

Multiplying t and $(1-t)$ on the both sides of (3.1) and (3.2), respectively yields that

$$g(u_m) - g(x_m) - \langle u_m, \nabla g(u_m)\rangle + \langle x, \nabla g(x)\rangle \ge \langle z, \nabla g(x) - \nabla g(u_m)\rangle.$$

That is, $D_g(z,x_m) \ge D_g(z,u_m)$, where $z \in C_m$. This finds that C_{m+1} is closed and convex. We conclude that C_n is closed and convex.

Next, we show that $A^{-1}(0) \cap EF(f,B,\varphi) \subset C_n$. $A^{-1}(0) \cap EF(f,B,\varphi) \subset C_1 = C$ is clear. Suppose that $A^{-1}(0) \cap EF(f,B,\varphi) \subset C_m$ for some positive integer m. For any $w \in CSS \subset C_m$, we see that

$$\begin{aligned}
D_g(w,u_m) &\le D_g(w,y_m) \\
&= D_g(w, \nabla g^*(\alpha_m \nabla g(x_m) + (1-\alpha_m)\nabla g J_s^A x_m)) \\
&= g(w) - \alpha_m \langle w, \nabla g(x_m)\rangle - (1-\alpha_m)\langle w, \nabla g(J_s^A x_m))\rangle \\
&\quad + g^*(\alpha_m \nabla g(x_m) + (1-\alpha_m)\nabla g(J_s^A x_m))) \\
&\le \alpha_m V(w, \nabla g(x_m)) + (1-\alpha_m) V(w, \nabla g(J_s^A x_m)) \\
&\le \alpha_m D_g(w,x_m) + (1-\alpha_m) D_g(w, J_s^A x_k) \\
&\le D_g(w,x_m),
\end{aligned} \tag{3.3}$$

which shows that $w \in C_{m+1}$. This implies that $A^{-1}(0) \cap EF(f) \subset C_n$. Notice that

$$\begin{aligned}
D_g(x_n,x_1) &\le D_g(Proj^g_{CSS} x_1, x_1) - D_g(Proj^g_{CSS} x_1, x_n) \\
&\le D_g(Proj^g_{CSS)} x_1, x_1).
\end{aligned}$$

This implies that the sequence $\{D_g(x_n,x_1)\}$ is bounded. It follows that the sequence $\{x_n\}$ is also bounded. In view of the construction of the sets C_n, we find that $C_m \subset C_n$ and $x_m = proj^g_{C_m} x_1 \in$

$C_m \subset C_n$ for any positive number $m \geq n$. It follows that

$$\begin{aligned} D_g(x_m, x_n) &= D_g(x_m, Proj^g_{C_n} x_1) \\ &\leq D_g(x_m, x_1) - D_g(Proj^g_{C_n} x_1, x_1) \\ &= D_g(x_m, x_1) - D_g(x_n, x_1). \end{aligned} \tag{3.4}$$

It follows that $D_g(x_n, x_1) \leq D_g(x_m, x_1)$. This shows that the sequence $\{D_g(x_n, x_1)\}$ is nondecreasing. Hence, the limit $\lim_{n\to\infty} D_g(x_n, x_1)$ exists. In view of $x_n = Proj^g_{C_n} x_1$, we find that $D_g(x_n, x_1) \leq D_g(x_{n+1}, x_1)$. This concludes that $\lim_{n\to\infty} D_g(x_n, x_1)$ exists. Let $m, n \to \infty$ in (3.4), we find that $D_g(x_m, x_n) \to 0$. Hence, we find that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. This yields that $\{x_n\}$ is a Cauchy sequence. Since C is closed and convex, we see that there exists an $\bar{x} \in C$ that

$$\lim_{n\to\infty} \|x_n - \bar{x}\| = 0. \tag{3.5}$$

Notice that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Since $x_{n+1} \in C_{n+1}$, we find that $D_g(x_{n+1}, u_n) \leq D_g(x_{n+1}, x_n)$. It follows that $\lim_{n\to\infty} D_g(x_{n+1}, u_n) = 0$. It follows that

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{3.6}$$

On the other hand, we have $\|x_n - u_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\|$. It follows from (3.6) that $\lim_{n\to\infty} \|x_n - u_n\| = 0$. This implies that

$$\lim_{n\to\infty} D_g(u_n, x_n) = 0. \tag{3.7}$$

Since ∇g is uniformly norm-to-norm continuous on any bounded subset of E, we find that

$$\lim_{n\to\infty} \|\nabla g(u_n) - \nabla g(x_n)\| = 0. \tag{3.8}$$

Notice that

$$\begin{aligned} |D_g(w, x_n) - D_g(w, u_n)| &= |D_g(u_n, x_n) + \langle w - u_n, \langle \nabla g(u_n) - \nabla g(x_n) \rangle| \\ &\leq |D_g(u_n, x_n)| + \|w - u_n\| \|\nabla g(u_n) - \nabla g(x_n)\|. \end{aligned}$$

It follows from (3.7) and (3.8) that

$$\lim_{n\to\infty} |D_g(w, x_n) - D_g(w, u_n)| = 0. \tag{3.9}$$

On the other hand, we have

$$\begin{aligned}D_g(w,u_n) &\le D_g(w,y_n)\\ &= V(w,\alpha_n\nabla g(x_n)+(1-\alpha_n)\nabla g(J_s^A x_n))\\ &= g(w)-\alpha_n\langle w,\nabla g(x_n)\rangle-(1-\alpha_n)\langle w,\nabla g(J_s^A x_n))\rangle\\ &\quad +g^*(\alpha_n\nabla g(x_n)+(1-\alpha_n)\nabla g(J_s^A x_n)))\\ &\le \alpha_n V(w,\nabla g(x_n))+(1-\alpha_n)V(w,\nabla g(J_s^A x_n))\\ &\quad -\rho(\|\nabla g(J_s^A x_n)-\nabla g(x_n)\|)\\ &\le D_g(w,x_n)-\alpha_n(1-\alpha_n)\rho(\|\nabla g(J_s^A x_n)-\nabla g(x_n)\|),\end{aligned}$$

where ρ is the gauge of the uniform convexity of g^*. It follows that

$$D_g(w,x_n)-D_g(w,u_n)\ge \alpha_n(1-\alpha_n)\rho(\|\nabla g(J_s^A x_n)-\nabla g(x_n)\|).$$

It follows that $\lim_{n\to\infty}\|\nabla g(x_n)-\nabla g(J_s^A x_n)\|=0$. Since ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we obtain that $\lim_{n\to\infty}\|x_n-J_s^A x_n\|=0$. Since J_s^A is closed Bregman quasi-nonexpansive, we find that $J_s^A\bar{x}=\bar{x}$.

On the other hand, we see that

$$D_g(w,x_n)-D_g(w,u_n)\ge D_g(w,y_n)-D_g(w,u_n)\ge D_g(u_n,y_n).$$

This implies that $\lim_{n\to\infty}D_g(u_n,y_n)=0$. Hence, we have $\lim_{n\to\infty}\|u_n-y_n\|=0$. Therefore, we have $\lim\limits_{n\to\infty}\|\nabla g(u_n)-\nabla(y_n)\|=0$. Without loss of generality, let us assume that there exists a real number a such that $r_n\ge a>0$ for all $n\ge 1$. Hence

$$\lim_{n\to\infty}\frac{\|\nabla g(u_n)-\nabla g(y_n)\|}{r_n}=0.$$

In view of $u_n=S_{r_n}y_n$, we see that $r_nF(u_n,y)+\langle y-u_n,\nabla g(u_n)-\nabla g(y_n)\rangle\ge 0,\quad \forall y\in C$. It follows from $(A2)$ that

$$\begin{aligned}\|y-u_n\|\frac{\|\nabla g(u_n)-\nabla g(y_n)\|}{r_n} &\ge \frac{1}{r_n}\langle y-u_n,\nabla g(u_n)-\nabla g(y_n)\rangle\\ &\ge F(y,u_n),\quad \forall y\in C.\end{aligned}$$

This implies that

$$F(y,\bar{x})\le 0,\quad \forall y\in C.$$

For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)\bar{x}$. It follows that $y_t \in C$, which yields that $F(y_t, \bar{x}) \le 0$. It follows from the $(A1)$ and $(A4)$ that

$$0 = F(y_t, y_t) \le tF(y_t, y) + (1-t)F(y_t, \bar{x}) \le tF(y_t, y).$$

That is, $F(y_t, y) \ge 0$. Letting $t_i \downarrow 0$, we obtain from $(A3)$ that $f(\bar{x}, y) \ge 0$, $\forall y \in C$. This implies that $\bar{x} \in EP(f, B, \varphi)$.

Finally, we prove that $\bar{x} = Proj^g_{A^{-1}(0) \cap EP(f,B,\varphi)} x_1$. In view of $x_n = Proj^g_{C_n} x_1$, we conclude that $\langle z - x_n, \nabla g(x_n) - \nabla g(x_1) \rangle, \forall z \in C_n$. Since $A^{-1}(0) \cap EP(f, B, \varphi) \subset C_n$, we arrive at

$$\langle w - x_n, \nabla g(x_n) - \nabla g(x_1) \rangle, \forall w \in A^{-1}(0) \cap EP(f, B, \varphi).$$

Letting $n \to \infty$ in the above inequality, we see that

$$\langle w - \bar{x}, \nabla g(\bar{x}) - \nabla g(x_1) \rangle \ge 0, \quad \forall w \in A^{-1}(0) \cap EP(f, B, \varphi).$$

This yields that $\bar{x} = Proj^g_{A^{-1}(0) \cap EP(f)} x_1$. This completes the proof.

From Theorem 3.1, we have the following results in Banach spaces.

Corollary 3.2. *Let E be a reflexive Banach space and let $g : E \to R$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of E. Let C be a nonempty, closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$ and let $A : E \to E^*$ be a maximal monotone operator such that $Dom(A) \subset C$. Let $B : C \to E^*$ be continuous and monotone mapping. Assume that the common solution set $A^{-1}(0) \cap EF(f, B)$ is nonempty. Let $\{x_n\}$ be a sequence generated* *in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = Proj^g_{C_1} x_0, \\ y_n = \nabla g^*(\alpha_n \nabla g(x_n) + (1-\alpha_n)\nabla g J^A_s x_n), \\ u_n \in C \textit{ such that } r_n f(u_n, y) + r_n \langle Bu_n, y - u_n\rangle + \langle y - u_n, \nabla g(u_n) - \nabla g(y_n)\rangle \geq 0, \quad \forall y \in C_n, \\ C_{n+1} = \{z \in C_n : D_g(z, u_n) \leq D_g(z, x_n)\}, \\ x_{n+1} = Proj^g_{C_{n+1}} x_1, \end{cases}$$

where $\{\alpha_n\}$ *is a real sequence in* $(0,1)$ *such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$*,* ∇g *is the right-hand derivative of* g*,* $J^A_s = (\nabla g + sA)^{-1}\nabla g$*, and* $\{r_n\}$ *is a real sequence such that* $\liminf_{n\to\infty} r_n > 0$*. Then the sequence* $\{x_n\}$ *converges strongly to* $Proj^g_{A^{-1}(0)\cap EF(f,B)} x_1$*, where* $Proj^g_{A^{-1}(0)\cap EF(f,B)}$ *is the Bregman projection from* E *onto* $A^{-1}(0) \cap EF(f,B)$*.*

Corollary 3.3. *Let* E *be a reflexive Banach space and let* $g : E \to R$ *be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of* E*. Let* C *be a nonempty, closed and convex subset of* E*. Let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* $(A1)$*-*$(A4)$ *and let* $A : E \to E^*$ *be a maximal monotone operator such that* $Dom(A) \subset C$*. Assume that the common solution set* $A^{-1}(0) \cap EF(f)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = Proj^g_{C_1} x_0, \\ y_n = \nabla g^*(\alpha_n \nabla g(x_n) + (1-\alpha_n)\nabla g J^A_s x_n), \\ u_n \in C \textit{ such that } r_n f(u_n, y) + \langle y - u_n, \nabla g(u_n) - \nabla g(y_n)\rangle \geq 0, \quad \forall y \in C_n, \\ C_{n+1} = \{z \in C_n : D_g(z, u_n) \leq D_g(z, x_n)\}, \\ x_{n+1} = Proj^g_{C_{n+1}} x_1, \end{cases}$$

where $\{\alpha_n\}$ *is a real sequence in* $(0,1)$ *such that* $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$*,* ∇g *is the right-hand derivative of g,* $J_s^A=(\nabla g+sA)^{-1}\nabla g$*, and* $\{r_n\}$ *is a real sequence such that* $\liminf_{n\to\infty} r_n>0$*. Then the sequence* $\{x_n\}$ *converges strongly to* $Proj^g_{A^{-1}(0)\cap EF(f)}x_1$*, where* $Proj^g_{A^{-1}(0)\cap EF(f)}$ *is the Bregman projection from E onto* $A^{-1}(0)\cap EF(f)$*.*

In the framework of Hilbert spaces, Corollary 3.3 is reduced to the following result.

Corollary 3.4. *Let E be a Hilbert space and let C be a nonempty, closed and convex subset of E. Let f be a bifunction from* $C\times C$ *to* $\mathbb{R}$ *satisfying* $(A1)$*-*$(A4)$ *and* $A:E\to E$ *be a maximal monotone operator such that* $Dom(A)\subset C$*. Assume that the common solution set* $A^{-1}(0)\cap EF(f)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0\in E \quad \text{chosen arbitrarily},\\ C_1=C,\\ x_1=Proj_{C_1}x_0,\\ y_n=\alpha_n x_n+(1-\alpha_n)J_s^A x_n),\\ u_n\in C \text{ such that } f(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-y_n\rangle\ge 0, \quad \forall y\in C,\\ C_{n+1}=\{z\in C_n:\|z-u_n\|^2\le\|z-x_n\|^2\},\\ x_{n+1}=Proj_{C_{n+1}}x_1, \end{cases}$$

where $\{\alpha_n\}$ *is a real sequence in* $(0,1)$ *such that* $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$*,* ∇g *is the right-hand derivative of g,* $J_s^A=(\nabla g+sA)^{-1}\nabla g$*, and* $\{r_n\}$ *is a real sequence such that* $\liminf_{n\to\infty} r_n>0$*. Then the sequence* $\{x_n\}$ *converges strongly to* $Proj^g_{A^{-1}(0)\cap EF(f)}x_1$*, where* $Proj^g_{A^{-1}(0)\cap EF(f)}$ *is the metric projection from E onto* $A^{-1}(0)\cap EF(f)$*.*

Proof. Putting $g(x)=\frac{1}{2}\|x\|^2$, we find that $\nabla g(x)=x$ for all $x\in E$. It follows that $D_g(x,y)=\|x-y\|^2$ for all $x,y\in E$. By Theorem 3.1, we can immediately obtain the desired conclusion.

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