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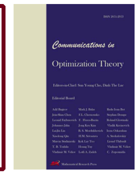

A SYSTEM OF NONCONVEX VARIATIONAL INEQUALITIES IN BANACH SPACES

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Abstract. The aim of this work is to suggest a new system of nonconvex variational inequalities in a q-uniformly smooth Banach space and establish an equivalence relation between this system and fixed point problems. By using the equivalence formulation, we construct a new perturbed projection iterative algorithm with mixed errors for finding a solution set of a system of nonconvex variational inequalities. Also, we prove the convergence theorem of the suggested iterative sequences generated by the algorithm.

Keywords. Nonconvex variational inequality; Iterative sequence; Convergence analysis; Mixed error; q-uniformly smooth Banach space.

1. Introduction

Variational inequalities were introduced by Stampacchia [1] provided us with a powerful tool to study a wide class of problems arising in mechanics, physics, optimization and control theory, linear programming, economics and engineering sciences, see [2, 3, 4, 5] and the reference therein.

In recent years, several authors studied different type of systems of variational inequalities and suggested iterative algorithms to find the approximate solutions of such systems, see [6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and the reference therein. We remark here that the almost

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all results concerning the system of solutions of iterative schemes for solving the system of variational inequalities and related problems are being considered in the setting of convex sets. Consequently the techniques are based on the projection of operators over convex sets, which may not hold in general, when the sets are nonconvex. It is known that the unified prox-regular sets are nonconvex and included the convex sets as special cases, see [16, 17, 18, 19, 20] and the reference therein.

Inspired by the recent work going on this fields, see for example, [21, 22, 23, 24, 25, 26, 27, 28], we introduced and studied a new system of nonconvex variational inequalities in q-uniformly smooth Banach spaces. We established the equivalence between the system of nonconvex variational inequalities and the fixed point problems. By using the equivalence formulation, we construct a perturbed projection iterative algorithm with mixed errors for finding a solution of the aforementioned system. Also, we prove the convergence of the iterative algorithms under suitable conditions.

2. Preliminaries

Let X be a real Banach space with the dual space X^* and let $\langle \cdot, \cdot \rangle$ be the dual pairing between X and X^*. Let $CB(X)$ denote the family of all nonempty closed bounded subset of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\}, \ \forall x \in X,$$

where $q > 1$ is a constant. In particular J_2 is a usual normalized duality mapping. It is known that in general

$$J_q(x) = \|x\|^{q-2} J_2(x)$$

for all $x \neq 0$ and J_q is single-valued if X^* is strictly convex. In the sequel, we always assume that X is a real Banach space such that J_q is a single valued. If X is a Hilbert space then J_q becomes the identity mapping on X.

The modulus of smoothness of X is the function $\rho_X : [0, \infty) \to [0, \infty)$ is defined by

$$\rho_X(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1, \|y\| \leq t\}.$$

A Banach space X is called uniformly smooth if

$$\lim_{t\to 0}\frac{\rho_X(t)}{t}=0.$$

X is called q-uniformly smooth $(q>1)$ if there exists a constant $c>0$ such that

$$\rho_X(t)<ct^q,\ \forall t>0.$$

Note that J_q is a single-valued if X is uniformly smooth. Concerned with the characteristic inequalities in q-uniformly smooth Banach spaces, Xu [29] proved the following results.

Lemma 2.1. [29] *The real Banach space X is q-uniformly smooth if and only if there exists a constant $c_q>0$ such that for all $x,y\in X$*

$$\|x+y\|^q\leq\|x\|^q+q\langle y,J_q(x)\rangle+c_q\|y\|^q.$$

For a nonempty closed subset K of X, we denote by $d_K(\cdot)$ or $d(\cdot,K)$ the usual distance function to K, that is,

$$d_K(u)=\inf_{v\in K}\|u-v\|.$$

Definition 2.2. The proximal normal cone of K at a point $u\in X-K$ is given by

$$N_K^P(u)=\{\zeta\in X: u\in P_K(u+\alpha\zeta)\text{ for some }\alpha>0\},$$

where $P_K(u)$ is the set of all projection of u onto K, that is,

$$P_K(u)=\{v\in K: d_K(u)=\|u-v\|\}.$$

Lemma 2.3. [3] *Let K be a nonempty closed subset of X. Then $\zeta\in N_K^P(u)$ if and only if there exists a constant $\alpha=\alpha(\zeta,u)>0$ such that*

$$\langle\zeta,v-u\rangle\leq\alpha\|v-u\|^2,\quad\forall v\in K.$$

Lemma 2.4. [3] *Let K be a nonempty closed and convex subset in X. Then $\zeta\in N_K^P(u)$ if and only if*

$$\langle\zeta,v-u\rangle\leq 0,\quad\forall v\in K.$$

Definition 2.5. Let $f : X \to \mathbb{R}$ be a locally Lipschitz continuous mapping with constant τ near a given point $x \in X$, that is, for some $\varepsilon > 0$, one has

$$| f(y) - f(z) | \leq \tau \|y - z\|, \quad \forall\, y, z \in B(x;\varepsilon),$$

where $B(x;\varepsilon)$ denotes the open ball of radius $r > 0$ and centered at x. The generalized directional derivative of f at x in the direction v, denoted by $f^o(x;v)$ is defined as follows

$$f^o(x;v) = \limsup_{y \to x_{t\downarrow 0}} \frac{f(y+tv) - f(y)}{t}$$

where y is a vector in X and t is a positive scalar.

Definition 2.6. The tangent cone $T_K(x)$ to K at a point $x \in K$ is defined as follows

$$T_K(x) = \{v \in X : d_K^o(x;v) = 0\}.$$

The normal cone of K at x by polarity with $T_K(x)$ is defined as follows

$$N_K(x) = \{\zeta : \langle \zeta, v\rangle \leq 0,\ \forall v \in T_K(x)\}.$$

The Clarke normal cone $N_K^C(x)$ is given by

$$N_K^C(x) = \overline{co}\left[N_K^P(x)\right],$$

where $\overline{co}(S)$ mean the closure of the convex hull of S.

It is clear that $N_K^P(x) \subseteq N_K^C(x)$. The converse is not true in general. Note that $N_K^C(x)$ is always closed and convex, where as $N_K^P(x)$ is always convex but may not be closed, see [3, 17, 30, 31, 32].

Definition 2.7. [30] For any $r \in (0, +\infty]$, a subset K_r of X is called the normalized uniformly prox-regular (or uniformly r-prox-regular) if every nonzero proximal normal to K_r can be realized by an r-ball, that is, for all $\bar{x} \in K_r$ and all $0 \neq \zeta \in N_{K_r}^P(\bar{x})$ with $\|\zeta\| = 1$,

$$\langle \zeta, x - \bar{x}\rangle \leq \frac{1}{2r}\|x - \bar{x}\|^2,\ \forall x \in K_r.$$

Lemma 2.8. [3] *A closed set $K \subseteq X$ is convex if and only if it is proximally smooth of radius r for every $r > 0$.*

Proposition 2.9. [32], *Let $r>0$ and let K_r be a nonempty closed and uniformly r-prox-regular subset of X. Set*

$$\mathscr{U}(r)=\{u\in X:0\leq d_{K_r}(u)<r\}.$$

Then the following statements are hold:

(a) *For all $x\in\mathscr{U}(r), P_{K_r}(x)\neq\emptyset$;*

(b) *For all $r'\in(0,r)$, P_{K_r} is a Lipschitz continuous mapping with constant $\frac{r}{r-r'}$ on*

$$\mathscr{U}(r')=\{u\in X:0\leq d_{K_r}(u)<r'\};$$

(c) *The proximal normal cone is closed as a set-valued mapping.*

From Proposition 2.9 (c) we have $N^C_{K_r}(x)=N^P_{K_r}(x)$. Therefore we define $N_{K_r}(x)=N^C_{K_r}(x)=N^P_{K_r}(x)$ for such a class of sets.

Definition 2.10. Let X be a q-uniformly smooth Banach space. A single-valued mapping $p:X\to X$ is called

(i) accretive if

$$\langle p(x)-p(y),j_q(x-y)\rangle\geq 0,\ \forall x,y\in X,$$

(ii) β-strongly accretive if there exists a constant $\beta>0$ such that

$$\langle p(x)-p(y),j_q(x-y)\rangle\geq\beta\|x-y\|^q,\ \forall x,y\in X,$$

(iii) relaxed ρ-strongly accretive if there exists a constant $\rho>0$ such that

$$\langle p(x)-p(y),j_q(x-y)\rangle\geq-\rho\|x-y\|^q,\ \forall x,y\in X,$$

(iv) σ-Lipschitz continuous if there exists a constant $\sigma>0$ such that

$$\|p(x)-p(y)\|\leq\sigma\|x-y\|,\ \forall x,y\in X.$$

Definition 2.11. Let $p:X\to X$ be a single-valued mapping and let $T:X\times X\to 2^{X^*}$ be a set valued mapping. Then T is said to be

(i) accretive if

$$\langle u-v,j_q(x-y)\rangle\geq 0,\ \forall x,x',y,y'\in X,u\in T(x,y),v\in T(x',y'),$$

(ii) (κ,υ)-relaxed cocoercive with respect to p if there exists a constant $(\kappa,\upsilon)>0$ such that for all $x,x',y,y'\in X$

$$\langle u-v, j_q(p(x)-p(y))\rangle \geq -\kappa\|u-v\|^q+\upsilon\|p(x)-p(y)\|^q,\ \forall u\in T(x,y), v\in T(x',y'),$$

(iii) $\xi-\widehat{\mathscr{D}}$-Lipschitz continuous in the first variable if there exists a constant $\xi>0$ such that for all $x,x'\in X$

$$\widehat{\mathscr{D}}(T(x,y),T(x',y'))\leq \xi\|x-x'\|,\ \forall y,y'\in X,$$

where $\widehat{\mathscr{D}}$ is the Hausdorff pseudo-metric, that is, for any two nonempty subsets A and B of X

$$\widehat{\mathscr{D}}(A,B)=\max\big\{\sup_{x\in A}d(x,B),\sup_{y\in B}d(y,A)\big\}.$$

3. A system of nonconvex variational inequalities

In this section, we introduce a new system of nonconvex variational inequalities in a Banach space and investigated their relations. Let $T_i: X\times X\to CB(X)$ be the nonlinear set valued mappings and let $g_i,h_i: X\to X$ be the nonlinear single valued mappings such that $K_r\subseteq g_i(X)$ for $i=1,2,3$. For any constants $\eta_i>0$ $(i=1,2,3)$, we consider the problem of finding $x,y,z\in X$ and $u\in T_1(y,x), v\in T_2(z,y), w\in T_3(x,z)$ such that $h_1(x),h_2(y),h_3(z)\in K_r, g_1(x^*),g_2(x^*),g_3(x^*)\in K_r$ and

$$\begin{aligned}\langle \eta_1 u+h_1(x)-g_1(y), g_1(x^*)-h_1(x)\rangle+\frac{1}{2r}\|g_1(x^*)-h_1(x)\|^2\geq 0,\ \forall x^*\in X,\\ \langle \eta_2 v+h_2(y)-g_2(z), g_2(x^*)-h_2(y)\rangle+\frac{1}{2r}\|g_2(x^*)-h_2(y)\|^2\geq 0,\ \forall x^*\in X,\\ \langle \eta_3 w+h_3(z)-g_3(x), g_3(x^*)-h_3(z)\rangle+\frac{1}{2r}\|g_3(x^*)-h_3(z)\|^2\geq 0,\ \forall x^*\in X.\end{aligned} \tag{3.1}$$

Problem (3.1) is called a system of nonconvex variational inequalities.

Lemma 3.1. *Let K_r be a uniformly r-prox-regular set. Then problem (3.1) is equivalent to finding $x,y,z\in X$ and $u\in T_1(y,x), v\in T_2(z,y), w\in T_3(x,z)$ such that $h_1(x),h_2(y),h_3(z)\in K_r$ and*

$$0\in \eta_1 u+h_1(x)-g_1(y)+N^P_{K_r}(h_1(x)),$$

$$0\in \eta_2 v+h_2(y)-g_2(z)+N^P_{K_r}(h_2(y)),$$

$$0 \in \eta_3 w + h_3(z) - g_3(x) + N^P_{K_r}(h_3(z)), \tag{3.2}$$

where $N^P_{K_r}(s)$ is the P-normal cone of K_r at s in the sense of nonconvex analysis.

Proof Let (x,y,z,u,v,w) with $x,y,z \in X, h_1(x), h_2(y), h_3(z) \in K_r$ and $u \in T_1(y,x), v \in T_2(z,y), w \in T_3(x,z)$ be a solution set of system (3.1). Since the vector zero always belongs to any normal cone, if

$$\eta_1 u + h_1(x) - g_1(y) = 0,$$

then

$$0 \in \eta_1 u + h_1(x) - g_1(y) + N^P_{K_r}(h_1(x)).$$

If

$$\eta_1 u + h_1(x) - g_1(y) \neq 0,$$

then for all $x^* \in X$ with $g_1(x^*) \in K_r$

$$\langle -(\eta_1 u + h_1(x) - g_1(y)), g_1(x^*) - h_1(x) \rangle \leq \frac{1}{2r} \|g_1(x^*) - h_1(x)\|^2. \tag{3.3}$$

From Lemma 2.3, we obtain

$$-(\eta_1 u + h_1(x) - g_1(y)) \in N^P_{K_r}(h_1(x)).$$

Hence, we have

$$0 \in \eta_1 u + h_1(x) - g_1(y) + N^P_{K_r}(h_1(x)). \tag{3.4}$$

Similarly, we have

$$0 \in \eta_2 v + h_2(y) - g_2(z) + N^P_{K_r}(h_2(y)) \tag{3.5}$$

and

$$0 \in \eta_3 w + h_3(z) - g_3(x) + N^P_{K_r}(h_3(z)). \tag{3.6}$$

Conversely, if (x,y,z,u,v,w) with $x,y,z \in X, h_1(x), h_2(y), h_3(z) \in K_r$ and $u \in T_1(y,x), v \in T_2(z,y), w \in T_3(x,z)$ is a solution set of system (3.2), then from Definition 2.7, (x,y,z,u,v,w) with $x,y,z \in X$ and $u \in T_1(y,x), v \in T_2(z,y), w \in T_3(x,z)$ and $h_1(x), h_2(y), h_3(z) \in K_r$ is a solution set of system (3.1).

The problem (3.2) is called a system of nonconvex variational inclusions.

4. Main results

Lemma 4.1. *For $i = 1,2,3$, let T_i, g_i, h_i, η_i be the same as in system (3.1). Then (x,y,z,u,v,w) with $x,y,z \in X, h_1(x), h_2(y), h_3(z) \in K_r$ and $u \in T_1(y,x), v \in T_2(z,y), w \in T_3(x,z)$ is a solution set of system (3.1) if and only if*

$$h_1(x) = P_{K_r}[g_1(y) - \eta_1 u],$$

$$h_2(y) = P_{K_r}[g_2(z) - \eta_2 v],$$

$$h_3(z) = P_{K_r}[g_3(x) - \eta_3 w], \tag{4.1}$$

where P_{K_r} is the projection of X onto the uniformly r-prox-regular set K_r.

Proof Let (x,y,z,u,v,w) with $x,y,z \in X, h_1(x), h_2(y), h_3(z) \in K_r$ and $u \in T_1(y,x)$, $v \in T_2(z,y)$, $w \in T_3(x,z)$ be a solution set of system (3.1). From Lemma 3.1, we have

$$0 \in \eta_1 u + h_1(x) - g_1(y) + N^P_{K_r}(h_1(x)),$$

$$0 \in \eta_2 v + h_2(y) - g_2(z) + N^P_{K_r}(h_2(y)),$$

$$0 \in \eta_3 w + h_3(z) - g_3(x) + N^P_{K_r}(h_3(z)), \tag{4.2}$$

which implies that

$$g_1(y) - \eta_1 u \in (I + N^P_{K_r})(h_1(x)),$$

$$g_2(z) - \eta_2 v \in (I + N^P_{K_r})(h_2(y)),$$

$$g_3(x) - \eta_3 w \in (I + N^P_{K_r})(h_3(z)). \tag{4.3}$$

Therefore, we have

$$h_1(x) = P_{K_r}[g_1(y) - \eta_1 u],$$

$$h_2(y) = P_{K_r}[g_2(z) - \eta_2 v],$$

$$h_3(z) = P_{K_r}[g_3(x) - \eta_3 w], \tag{4.4}$$

where I is an identity mapping and $P_{K_r} = (I + N^P_{K_r})^{-1}$. The converse part of this lemma is also trivial. This completes the proof.

Remark 4.2. Inequality (4.1) can be written as follows:

$$p = g_1(y) - \eta_1 u,\ h_1(x) = P_{K_r}(p),$$

$$q = g_2(z) - \eta_2 v,\ h_2(y) = P_{K_r}(q),$$

$$t = g_3(x) - \eta_3 w,\ h_3(z) = P_{K_r}(t), \tag{4.5}$$

where $\eta_i > 0, i = 1,2,3$ are constants.

The fixed point formulation (4.5) enables us to construct the following perturbed iterative algorithm with mixed errors.

Algorithm 4.3. *Let $T_i, g_i, h_i, \eta_i > 0, i = 1,2,3$ be the same as in system (3.1) such that $h_i : X \to X$ is an onto operator for each $i = 1,2,3$. For arbitrary chosen initial points $(p_0, q_0, t_0) \in X \times X \times X$, compute the iterative sequences $\{(x_n, y_n, z_n, u_n, v_n, w_n)\}_{n=0}^{\infty}$ by using*

$$h_1(x_n) = P_{K_r}(p_n),\ p_{n+1} = (1-\alpha_n)p_n + \alpha_n(g_1(y_n) - \eta_1 u_n + e_n) + r_n,$$

$$h_2(y_n) = P_{K_r}(q_n),\ q_{n+1} = (1-\alpha_n)q_n + \alpha_n(g_2(z_n) - \eta_2 v_n + c_n) + s_n,$$

$$h_3(z_n) = P_{K_r}(t_n),\ t_{n+1} = (1-\alpha_n)t_n + \alpha_n(g_3(x_n) - \eta_3 w_n + d_n) + \ell_n, \tag{4.6}$$

and

$$u_n \in T_1(y_n, x_n); \|u_n - u_{n+1}\| \le (1 + (1+n)^{-1})\widehat{\mathscr{D}}(T_1(y_n, x_n), T_1(y_{n+1}, x_{n+1})),$$

$$v_n \in T_2(z_n, y_n); \|v_n - v_{n+1}\| \le (1 + (1+n)^{-1})\widehat{\mathscr{D}}(T_2(z_n, y_n), T_2(z_{n+1}, y_{n+1})),$$

$$w_n \in T_3(x_n, z_n); \|w_n - w_{n+1}\| \le (1 + (1+n)^{-1})\widehat{\mathscr{D}}(T_3(x_n, z_n), T_3(x_{n+1}, z_{n+1})),$$

where initial points $u_0 \in T_1(y_0, x_0), v_0 \in T_2(z_0, y_0), w_0 \in T_3(x_0, z_0)$ are chosen arbitrary, $0 < \alpha_n \le 1$ is a parameter and $\{e_n\}_{n=0}^{\infty}, \{c_n\}_{n=0}^{\infty}, \{d_n\}_{n=0}^{\infty}, \{r_n\}_{n=0}^{\infty}, \{s_n\}_{n=0}^{\infty}$ and $\{\ell_n\}_{n=0}^{\infty}$ are six sequences in X to take into account of a possible inexact computation of the resolvent operator satisfying the following conditions:

$$\lim_{n\to\infty} e_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = \lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \ell_n = 0,$$

$$\sum_{n=1}^{\infty} \|e_n - e_{n-1}\| < \infty,\ \sum_{n=1}^{\infty} \|c_n - c_{n-1}\| < \infty,\ \sum_{n=1}^{\infty} \|d_n - d_{n-1}\| < \infty,$$

$$\sum_{n=1}^{\infty} \|r_n - r_{n-1}\| < \infty,\ \sum_{n=1}^{\infty} \|s_n - s_{n-1}\| < \infty,\ \sum_{n=1}^{\infty} \|\ell_n - \ell_{n-1}\| < \infty. \tag{4.7}$$

Theorem 4.4. *Let $T_i, g_i, h_i, \eta_i, i = 1,2,3$ be the same as in system (3.1) such that*

(i) *T_i is a relaxed (κ_i, υ_i)-cocoercive with constant $\kappa_i, \upsilon_i > 0$ and $\xi_i - \widehat{\mathscr{D}}$-Lipschitz continuous mapping in the first variable,*

(ii) *h_i is a β_i-strongly accretive and σ_i-Lipschitz continuous mapping,*

(iii) *g_i is a μ_i-Lipschitz continuous and relaxed ρ_i-strongly accretive mapping.*

If the constants $\eta_i > 0$ satisfying the following conditions:

$$\frac{r(\Omega_1+\varphi_1)}{(r-r')(1-\pi_2)} < 1,\ \frac{r(\Omega_2+\varphi_2)}{(r-r')(1-\pi_3)} < 1,\ \frac{r(\Omega_3+\varphi_3)}{(r-r')(1-\pi_1)} < 1, \tag{4.8}$$

$$\frac{\Omega_1+\varphi_1}{1-\pi_2} < 1,\ \frac{\Omega_2+\varphi_2}{1-\pi_3} < 1,\ \frac{\Omega_3+\varphi_3}{1-\pi_1} < 1,$$

for each $i = 1,2,3$

$$\varphi_i = \sqrt[q]{1-q\eta_i(\kappa_i\xi_i^q-\upsilon_i)+c_q\eta_i^q\xi_i^q},\ \Omega_i = \sqrt[q]{1+q\rho_i+c_q\mu_i^q}$$

$$\pi_i = \sqrt[q]{1-q\beta_i+c_q\sigma_i^q} < 1,$$

where $r' \in (0,r)$, then there exists $x^*, y^*, z^* \in X$ with $h_1(x^*), h_2(y^*), h_3(z^*) \in K_r$ and $u^* \in T_1(y^*,x^*)$, $v^* \in T_2(z^*,y^*)$, $w^* \in T_3(x^*,z^*)$ such that $(x^*,y^*,z^*,u^*,v^*,w^*)$ is a solution of (3.1) and sequences $\{(x_n,y_n,z_n,u_n,v_n,w_n)\}_{n=0}^{\infty}$ generated by Algorithm 4.3 converges strongly to $(x^*,y^*,z^*,u^*,v^*,w^*)$, respectively.

Proof. From (4.6), we have

$$\begin{aligned}\|p_{n+1}-p_n\| &\leq (1-\alpha_n)\|p_n-p_{n-1}\| + \alpha_n(\|y_n-y_{n-1}-(g_1(y_n)-g_1(y_{n-1}))\| \\ &\quad + \|y_n-y_{n-1}-\eta_1(u_n-u_{n-1})\| + \|e_n-e_{n-1}\|) + \|r_n-r_{n-1}\|. \end{aligned} \tag{4.9}$$

Since g_1 is a relaxed ρ_1-strongly accretive and μ_1-Lipschitz continuous mapping, we have

$$\begin{aligned}&\|y_n-y_{n-1}-(g_1(y_n)-g_1(y_{n-1}))\|^q \\ &= \|y_n-y_{n-1}\|^q - q\langle g_1(y_n)-g_1(y_{n-1}), j_q(y_n-y_{n-1})\rangle + c_q\|g_1(y_n)-g_1(y_{n-1})\|^q \\ &\leq \|y_n-y_{n-1}\|^q + q\rho_1\|y_n-y_{n-1}\|^q + \mu_1^q c_q\|y_n-y_{n-1}\|^q \\ &\leq (1+q\rho_1+\mu_1^q c_q)\|y_n-y_{n-1}\|^q.\end{aligned}$$

Therefore, we have

$$\|y_n - y_{n-1} - (g_1(y_n) - g_1(y_{n-1}))\| \leq \sqrt[q]{1 + q\rho_1 + c_q\mu_1^q}\|y_n - y_{n-1}\|. \tag{4.10}$$

Since T_1 is a $\xi_1 - \widehat{\mathscr{D}}$-Lipschitz continuous mapping with constant $\xi_1 > 0$, we have

$$\|u_n - u_{n-1}\| \leq \xi_1(1+n^{-1})\widehat{\mathscr{D}}(T_1(y_n,x_n),T_1(y_{n-1},x_{n-1})) \leq \xi_1(1+n^{-1})\|y_n - y_{n-1}\|. \tag{4.11}$$

Again T_1 is a relaxed (κ_1, υ_1)-cocoercive and $\xi_1 - \widehat{\mathscr{D}}$-Lipschitz continuous mapping in the first variable, we get

$$\begin{aligned}
&\|y_n - y_{n-1} - \eta_1(u_n - u_{n-1})\|^q \\
&\leq \|y_n - y_{n-1}\|^q - q\eta_1\langle u_n - u_{n-1}, j_q(y_n - y_{n-1})\rangle + c_q\eta_1^q\|u_n - u_{n-1}\|^q \\
&\leq \|y_n - y_{n-1}\|^q - q\eta_1(-\kappa_1\|u_n - u_{n-1}\|^q + \upsilon_1\|y_n - y_{n-1}\|^q) \\
&\quad + c_q\eta_1^q\xi_1^q(1+n^{-1})^q\|y_n - y_{n-1}\|^q \\
&\leq \|y_n - y_{n-1}\|^q + (q\eta_1\kappa_1\xi_1^q(1+n^{-1})^q - q\eta_1\upsilon_1)\|y_n - y_{n-1}\|^q \\
&\quad + c_q\eta_1^q\xi_1^q(1+n^{-1})^q\|y_n - y_{n-1}\|^q \\
&\leq (1 - q\eta_1(\kappa_1\xi_1^q(1+n^{-1})^q - \upsilon_1) + c_q\eta_1^q\xi_1^q(1+n^{-1})^q)\|y_n - y_{n-1}\|^q.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\|y_n - y_{n-1} - \eta_1(u_n - u_{n-1})\| \\
&\leq \sqrt[q]{1 - q\eta_1(\kappa_1\xi_1^q(1+n^{-1})^q - \upsilon_1) + c_q\eta_1^q\xi_1^q(1+n^{-1})^q}\|y_n - y_{n-1}\|.
\end{aligned} \tag{4.12}$$

From (4.9), (4.10) and (4.12), we get

$$\begin{aligned}
\|p_{n+1} - p_n\| \leq (1-\alpha_n)\|p_n - p_{n-1}\| + \alpha_n\Bigg(&\sqrt[q]{1 + q\rho_1 + c_q\mu_1^q} \\
&+ \sqrt[q]{1 - q\eta_1(\kappa_1\xi_1^q(1+n^{-1})^q - \upsilon_1) + c_q\eta_1^q\xi_1^q(1+n^{-1})^q}\Bigg)\|y_n - y_{n-1}\| \\
&+ \alpha_n\|e_n - e_{n-1}\| + \|r_n - r_{n-1}\|.
\end{aligned} \tag{4.13}$$

Using the similar computation from (4.9)-(4.13), we have

$$\begin{aligned}
\|q_{n+1} - q_n\| \leq (1-\alpha_n)\|q_n - q_{n-1}\| + \alpha_n\Bigg(&\sqrt[q]{1 + q\rho_2 + c_q\mu_2^q} \\
&+ \sqrt[q]{1 - q\eta_2(\kappa_2\xi_2^q(1+n^{-1})^q - \upsilon_2) + c_q\eta_2^q\xi_2^q(1+n^{-1})^q}\Bigg)\|z_n - z_{n-1}\|
\end{aligned}$$

$$+\alpha_n\|c_n - c_{n-1}\| + \|s_n - s_{n-1}\|. \tag{4.14}$$

Again using the similar computation from (4.9)-(4.14), we have

$$\|t_{n+1} - t_n\| \le (1-\alpha_n)\|t_n - t_{n-1}\| + \alpha_n\Bigg(\sqrt[q]{1+q\rho_3+c_q\mu_3^q}$$

$$+\sqrt[q]{1-q\eta_3(\kappa_3\xi_3^q(1+n^{-1})^q - \upsilon_3) + c_q\eta_3^q\xi_3^q(1+n^{-1})^q}\Bigg)\|x_n - x_{n-1}\|$$

$$+\alpha_n\|d_n - d_{n-1}\| + \|\ell_n - \ell_{n-1}\|. \tag{4.15}$$

On the other hand, by using (4.6) and Proposition 2.9, we find that

$$\begin{aligned}\|x_n - x_{n-1}\| &\le \|x_n - x_{n-1} - (h_1(x_n) - h_1(x_{n-1}))\| + \|h_1(x_n) - h_1(x_{n-1})\| \\ &\le \|x_n - x_{n-1} - (h_1(x_n) - h_1(x_{n-1}))\| + \|P_{K_r}(p_n) - P_{K_r}(p_{n-1})\| \\ &\le \|x_n - x_{n-1} - (h_1(x_n) - h_1(x_{n-1}))\| + \frac{r}{r-r'}\|p_n - p_{n-1}\|.\end{aligned} \tag{4.16}$$

Since h_1 is β_1-strongly accretive and σ_1-Lipschitz continuous, we have

$$\begin{aligned}&\|x_n - x_{n-1} - (h_1(x_n) - h_1(x_{n-1}))\|^q \\ &= \|x_n - x_{n-1}\|^q - q\langle h_1(x_n) - h_1(x_{n-1}), j_q(x_n - x_{n-1})\rangle + c_q\|h_1(x_n) - h_1(x_{n-1})\|^q \\ &\le \|x_n - x_{n-1}\|^q - q\beta_1\|x_n - x_{n-1}\|^q + c_q\sigma_1^q\|x_n - x_{n-1}\|^q \\ &\le (1 - q\beta_1 + c_q\sigma_1^q)\|x_n - x_{n-1}\|^q.\end{aligned}$$

Therefore, we have

$$\|x_n - x_{n-1} - (h_1(x_n) - h_1(x_{n-1}))\| \le \sqrt[q]{1 - q\beta_1 + c_q\sigma_1^q}\|x_n - x_{n-1}\|. \tag{4.17}$$

Substituting (4.17) in (4.16), we obtain

$$\|x_n - x_{n-1}\| \le \sqrt[q]{1 - q\beta_1 + c_q\sigma_1^q}\|x_n - x_{n-1}\| + \frac{r}{r-r'}\|p_n - p_{n-1}\|, \tag{4.18}$$

which implies that

$$\|x_n - x_{n-1}\| \le \frac{r}{(r-r')(1 - \sqrt[q]{1 - q\beta_1 + c_q\sigma_1^q})}\|p_n - p_{n-1}\|. \tag{4.19}$$

Similarly, we have

$$\|y_n - y_{n-1}\| \le \frac{r}{(r-r')(1 - \sqrt[q]{1 - q\beta_2 + c_q\sigma_2^q})}\|q_n - q_{n-1}\|, \tag{4.20}$$

and

$$\|z_n - z_{n-1}\| \leq \frac{r}{(r-r')(1-\sqrt[q]{1-q\beta_3+c_q\sigma_3^q})}\|t_n - t_{n-1}\|. \tag{4.21}$$

It follows from (4.13) and (4.20) that

$$\|p_{n+1} - p_n\| \leq (1-\alpha_n)\|p_n - p_{n-1}\| + \alpha_n \frac{r(\Omega_1+\varphi_1)}{(r-r')(1-\pi_2)}\|q_n - q_{n-1}\|$$
$$+\alpha_n\|e_n - e_{n-1}\| + \|r_n - r_{n-1}\|, \tag{4.22}$$

where

$$\varphi_1 = \sqrt[q]{1-q\eta_1(\kappa_1\xi_1^q(1+n^{-1})^q - \upsilon_1) + c_q\eta_1^q\xi_1^q(1+n^{-1})^q},$$
$$\Omega_1 = \sqrt[q]{1+q\rho_1+c_q\mu_1^q}$$

and

$$\pi_2 = \sqrt[q]{1-q\beta_2+c_q\sigma_2^q}.$$

From (4.14) and (4.21), we obtain

$$\|q_{n+1} - q_n\| \leq (1-\alpha_n)\|q_n - q_{n-1}\| + \alpha_n \frac{r(\Omega_2+\varphi_2)}{(r-r')(1-\pi_3)}\|t_n - t_{n-1}\|$$
$$+\alpha_n\|c_n - c_{n-1}\| + \|s_n - s_{n-1}\|, \tag{4.23}$$

where

$$\varphi_2 = \sqrt[q]{1-q\eta_2(\kappa_2\xi_2^q(1+n^{-1})^q - \upsilon_2) + c_q\eta_2^q\xi_2^q(1+n^{-1})^q},$$
$$\Omega_2 = \sqrt[q]{1+q\rho_2+c_q\mu_2^q}$$

and

$$\pi_3 = \sqrt[q]{1-q\beta_3+c_q\sigma_3^q}.$$

Again from (4.15) and (4.19), we obtain

$$\|t_{n+1} - t_n\| \leq (1-\alpha_n)\|t_n - t_{n-1}\| + \alpha_n \frac{r(\Omega_3+\varphi_3)}{(r-r')(1-\pi_1)}\|p_n - p_{n-1}\|$$
$$+\alpha_n\|d_n - d_{n-1}\| + \|\ell_n - \ell_{n-1}\|, \tag{4.24}$$

where

$$\varphi_3 = \sqrt[q]{1-q\eta_3(\kappa_3\xi_3^q(1+n^{-1})^q - \upsilon_3) + c_q\eta_3^q\xi_3^q(1+n^{-1})^q},$$
$$\Omega_3 = \sqrt[q]{1+q\rho_3+c_q\mu_3^q}$$

and

$$\pi_1 = \sqrt[q]{1 - q\beta_1 + c_q\sigma_1^q}.$$

Now we define $\|\cdot\|_*$ on $X \times X \times X$ by

$$\|(x,y,z)\|_* = \|x\|_* + \|y\|_* + \|z\|_* \,, \forall (x,y,z) \in X \times X \times X.$$

It is obvious that $(X \times X \times X, \|\cdot\|_*)$ is a Banach space. Applying (4.22)-(4.24), we have

$$\begin{aligned}
&\|(p_{n+1},q_{n+1},t_{n+1}) - (p_n,q_n,t_n)\|_* \\
&= (1-\alpha_n)\|(p_n,q_n,t_n) - (p_{n-1},q_{n-1},t_{n-1})\|_* + \alpha_n \chi(n)\|(p_n,q_n,t_n) - (p_{n-1},q_{n-1},t_{n-1})\|_* \\
&\quad + \alpha_n\|(e_n,c_n,d_n) - (e_{n-1},c_{n-1},d_{n-1})\|_* + \|(r_n,s_n,\ell_n) - (r_{n-1},s_{n-1},\ell_{n-1})\|_*, \qquad (4.25)
\end{aligned}$$

where

$$\chi(n) = \max\left\{\frac{r(\Omega_1+\varphi_1)}{(r-r')(1-\pi_2)}, \frac{r(\Omega_2+\varphi_2)}{(r-r')(1-\pi_3)}, \frac{r(\Omega_3+\varphi_3)}{(r-r')(1-\pi_1)}\right\}. \qquad (4.26)$$

Let $\chi(n) \to \chi$ as $n \to \infty$, where

$$\chi = \max\left\{\frac{r(\Omega_1+\varphi_1)}{(r-r')(1-\pi_2)}, \frac{r(\Omega_2+\varphi_2)}{(r-r')(1-\pi_3)}, \frac{r(\Omega_3+\varphi_3)}{(r-r')(1-\pi_1)}\right\}. \qquad (4.27)$$

From condition (4.8), we know that $0 \le \chi < 1$. Then for $\widehat{\chi} = \frac{1}{2}(\chi+1) \in (\chi,1)$ there exists $n_0 \ge 1$ such that $\chi(n) = \widehat{\chi}$ for each $n \ge n_0$. Thus it follows from (4.25) that for each $n \ge n_0$,

$$\begin{aligned}
&\|(p_{n+1},q_{n+1},t_{n+1}) - (p_n,q_n,t_n)\|_* \\
&= (1-\alpha_n)\|(p_n,q_n,t_n) - (p_{n-1},q_{n-1},t_{n-1})\|_* + \alpha_n\widehat{\chi}\|(p_n,q_n,t_n) - (p_{n-1},q_{n-1},t_{n-1})\|_* \\
&\quad + \alpha_n\|(e_n,c_n,d_n) - (e_{n-1},c_{n-1},d_{n-1})\|_* + \|(r_n,s_n,\ell_n) - (r_{n-1},s_{n-1},\ell_{n-1})\|_* \\
&\le (1-\alpha_n(1-\widehat{\chi}))\big[(1-\alpha_n(1-\widehat{\chi}))\|(p_{n-1},q_{n-1},t_{n-1}) - (p_{n-2},q_{n-2},t_{n-2})\|_* \\
&\quad + \alpha_n\|(e_{n-1},c_{n-1},d_{n-1}) - (e_{n-2},c_{n-2},d_{n-2})\|_* + \|(r_{n-1},s_{n-1},\ell_{n-1}) - (r_{n-2},s_{n-2},\ell_{n-2})\|_*\big] \\
&\quad + \alpha_n\|(e_n,c_n,d_n) - (e_{n-1},c_{n-1},d_{n-1})\|_* + \|(r_n,s_n,\ell_n) - (r_{n-1},s_{n-1},\ell_{n-1})\|_* \\
&= (1-\alpha_n(1-\widehat{\chi}))^2\|(p_{n-1},q_{n-1},t_{n-1}) - (p_{n-2},q_{n-2},t_{n-2})\|_* \\
&\quad + \alpha_n\big[(1-\alpha_n(1-\widehat{\chi}))\|(e_{n-1},c_{n-1},d_{n-1}) - (e_{n-2},c_{n-2},d_{n-2})\|_* \\
&\quad + \|(e_n,c_n,d_n) - (e_{n-1},c_{n-1},d_{n-1})\|_*\big]
\end{aligned}$$

$$+(1-\alpha_n(1-\widehat{\chi}))\|(r_{n-1},s_{n-1},\ell_{n-1})-(r_{n-2},s_{n-2},\ell_{n-2})\|_*$$

$$+\|(r_n,s_n,\ell_n)-(r_{n-1},s_{n-1},\ell_{n-1})\|_*$$

$$\leq$$

$$\vdots$$

$$\leq (1-\alpha_n(1-\widehat{\chi}))^{n-n_0}\|(p_{n_0+1},q_{n_0+1},t_{n_0+1})-(p_{n_0},q_{n_0},t_{n_0})\|_*$$

$$+\alpha_n\sum_{i=1}^{n-n_0}(1-\alpha_n(1-\widehat{\chi}))^{i=1}\|(e_{n-(i-1)},c_{n-(i-1)},d_{n-(i-1)})-(e_{n-i},c_{n-i},d_{n-i})\|_*$$

$$+\sum_{i=1}^{n-n_0}(1-\alpha_n(1-\widehat{\chi}))^{i-1}\|(r_{n-(i-1)},s_{n-(i-1)},\ell_{n-(i-1)})-(r_{n-i},s_{n-i},\ell_{n-i})\|_*. \tag{4.28}$$

Hence for any $m\geq n>n_0$, we have

$$\|(p_m,q_m,t_m)-(p_n,q_n,t_n)\|_*\leq\sum_{j=n}^{m-1}\|(p_{j+1},q_{j+1},t_{j+1})-(p_j,q_j,t_j)\|_*$$

$$\leq\sum_{j=n}^{m-1}(1-\alpha_n(1-\widehat{\chi}))^{j-n_0}\|(p_{n_0+1},q_{n_0+1},t_{n_0+1})-(p_{n_0},q_{n_0},t_{n_0})\|_*$$

$$+\alpha_n\sum_{j=n}^{m-1}\sum_{i=1}^{j-n_0}(1-\alpha_n(1-\widehat{\chi}))^{i=1}\|(e_{n-(i-1)},c_{n-(i-1)},d_{n-(i-1)})-(e_{n-i},c_{n-i},d_{n-i})\|_*$$

$$+\sum_{j=n}^{m-1}\sum_{i=1}^{j-n_0}(1-\alpha_n(1-\widehat{\chi}))^{i=1}\|(r_{n-(i-1)},s_{n-(i-1)},\ell_{n-(i-1)})-(r_{n-i},s_{n-i},\ell_{n-i})\|_*. \tag{4.29}$$

Since $(1-\alpha_n(1-\widehat{\chi}))\in(0,1)$, it follows from (4.7) and (4.29) that

$$\|(p_m,q_m,t_m)-(p_n,q_n,t_n)\|_*=\|p_m-p_n\|_*+\|q_m-q_n\|_*+\|t_m-t_n\|_*\to 0 \text{ as } n\to\infty.$$

Hence $\{p_n\},\{q_n\}$ and $\{t_n\}$ are Cauchy sequences in X and so there exists p^*,q^* and $t^*\in X$ such that $p_n\to p^*,q_n\to q^*$ and $t_n\to t^*$ as $n\to\infty$. By the inequalities (4.19), (4.20), (4.21), it follows that the sequences $\{x_n\},\{y_n\}$ and $\{z_n\}$ are Cauchy sequences in X. Thus there exists $x^*,y^*,z^*\in X$ such that $x_n\to x^*,y_n\to y^*$ and $z_n\to z^*$ as $n\to\infty$. Since for each $i=1,2,3,T_i$ is a

$\xi_i - \widehat{\mathscr{D}}$-Lipschitz continuous mapping in the first variable, therefore it follows from (4.6) that

$$\begin{cases} \|u_n - u_{n+1}\| & \leq (1+(1+n)^{-1})\widehat{\mathscr{D}}(T_1(y_n,x_n),T_1(y_{n+1},x_{n+1})) \\ & \leq (1+(1+n^{-1}))\xi_1\|y_n - y_{n+1}\| \to 0, \text{ as } n\to\infty, \\ \|v_n - v_{n+1}\| & \leq (1+(1+n)^{-1})\widehat{\mathscr{D}}(T_2(z_n,y_n),T_2(z_{n+1},y_{n+1})) \\ & \leq (1+(1+n^{-1}))\xi_2\|z_n - z_{n+1}\| \to 0, \text{ as } n\to\infty, \\ \|w_n - w_{n+1}\| & \leq (1+(1+n)^{-1})\widehat{\mathscr{D}}(T_3(x_n,z_n),T_3(x_{n+1},z_{n+1})) \\ & \leq (1+(1+n^{-1}))\xi_3\|x_n - x_{n+1}\| \to 0, \text{ as } n\to\infty. \end{cases} \tag{4.30}$$

Hence $\{u_n\},\{v_n\},\{w_n\}$ are Cauchy sequences in X and so there exists $u^*,v^*,w^* \in X$ such that $x_n \to x^*, y_n \to y^*$ and $z_n \to z^*$ as $n\to\infty$. Further $u_n \in T_1(y_n,x_n)$ we have

$$\begin{aligned} d(u^*,T_1(y^*,x^*)) &= \inf\{\|u^* - \varsigma\| : \varsigma \in T_1(y^*,x^*)\} \\ &\leq \|u^* - u_n\| + d(u_n,T_1(y^*,x^*)) \\ &\leq \|u^* - u_n\| + (1+n^{-1})\widehat{\mathscr{D}}(T_1(y_n,x_n),T_1(y^*,x^*)) \\ &\leq \|u^* - u_n\| + \xi_1\|y_n - y^*\| \to 0 \text{ as } n\to\infty. \end{aligned} \tag{4.31}$$

Hence $d(u^*,T_1(y^*,x^*)) = 0$, therefore $u_n \to u^* \in T_1(y^*,x^*) \in X$. Since $v_n \in T_2(z_n,y_n)$, by the similar proof of (4.31), we obtain

$$d(v^*,T_2(z^*,y^*)) \leq \|v^* - v_n\| + \xi_2\|z_n - z^*\| \to 0 \text{ as } n\to\infty. \tag{4.32}$$

Again, since $w_n \in T_3(x_n,z_n)$, we obtain

$$d(w^*,T_3(x^*,z^*)) \leq \|w^* - w_n\| + \xi_3\|x_n - x^*\| \to 0 \text{ as } n\to\infty. \tag{4.33}$$

The right side of inequalities (4.31), (4.32) and (4.33) tend to zero as $n\to\infty$. Hence $u^* \in T_1(y^*,x^*), v^* \in T_2(z^*,y^*), w^* \in T_3(x^*,z^*)$. Since the operators g_1,g_2 and g_3 are continuous, it follows from (4.6) and (4.7) that

$$p^* = g_1(y^*) - \eta_1 u^*,\ q^* = g_2(z^*) - \eta_2 v^*,\ t^* = g_3(x^*) - \eta_3 w^*. \tag{4.34}$$

Since the operators h_1, h_2, h_3 and P_{K_r} are continuous mappings, it follows from (4.6) and (4.34) that

$$h_1(x^*) = P_{K_r}(p^*) = P_{K_r}(g_1(y^*) - \eta_1 u^*),$$

$$h_2(y^*) = P_{K_r}(q^*) = P_{K_r}(g_2(z^*) - \eta_2 v^*), \tag{4.35}$$

$$h_3(z^*) = P_{K_r}(t^*) = P_{K_r}(g_3(x^*) - \eta_3 w^*).$$

In view of Lemma 4.1, we see that it grantees that $(x^*, y^*, z^*, u^*, v^*, w^*)$ is a solution set of system (3.1). This completes the proof.

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