

Communications in Optimization Theory Available online at http://cot.mathres.org

## FIXED POINT THEOREMS FOR GENERALIZED $F - \varphi$ – WEAK CONTRACTIONS IN COMPLETE METRIC SPACES

## HOSSEIN PIRI

Department of Mathematics, University of Bonab, Bonab 5551761167, Iran

Abstract. The aim of this paper is to establish some new fixed point results for generalized  $F - \varphi$  – weak contractions in complete metric spaces. Our results improve and extend the corresponding results announced by many others.

**Keywords:** Fixed point; Metric space; Generalized  $F - \varphi$  – weak contraction.

## 1. Introduction and preliminaries

Throughout this article, we denote by  $\mathbb{R}$  the set of all real numbers, by  $\mathbb{R}_+$  the set of all positive real numbers, by  $\mathbb{N}_0$  the set of all nonnegative integer numbers and by  $\mathbb{N}$  the set of all natural numbers.

Let (X,d) be a metric space. A map  $T: X \to X$  is a contraction if for each  $x, y \in X$ , there exists a constant  $k \in (0,1)$  such that  $d(Tx,Ty) \le kd(x,y)$ .

A map  $T: X \to X$  is a  $\varphi$ -weak contraction if for each  $x, y \in X$ , there exists a function  $\varphi: [0, \infty) \to [0, \infty)$  such that  $\varphi$  is positive on  $(0, \infty)$  and  $\varphi(0) = 0$ , and  $d(Tx, Ty) \le d(x, y) - \varphi(d(x, y))$ .

The concept of weak contraction is introduced by Alber and Guerre-Delabriere [1]. They proved the existence of fixed points for single-valued maps satisfying weak contractive conditions on Hilbert spaces. Rhoades [7] showed that the result of Alber et al. is also valid in complete metric spaces. He proved the following very interesting fixed point theorem which is one of generalizations of the Banach contraction principle because it contains contractions as special cases

$$\boldsymbol{\varphi}(t) = (1-k)t.$$

E-mail addresses: h.piri@bonabu.ac.ir; hossein\_piri1979@yahoo.com Received August 14, 2015

**Theorem 1.1.** [7] Let (X, d) be a complete metric space and let  $T : X \to X$  be a weakly contraction on X. If  $\varphi$  is a continuous and nondecreasing function with  $\varphi(t) > 0$  for  $t \in (0, \infty)$  and  $\varphi(0) = 0$ , then T has a unique fixed point  $x^* \in X$ .

In 2012, Wardowski [9] introduced and studied a new contraction called F-contraction to prove a fixed point result as a generalization of the Banach contraction principle.

**Definition 1.1.** Let  $\mathscr{F}$  be the family of all functions  $F : \mathbb{R}_+ \to \mathbb{R}$  such that

- (*F*1) *F* is strictly increasing, i.e. for all  $x, y \in \mathbb{R}_+$  such that x < y, F(x) < F(y);
- (F2) For each sequence  $\{\alpha_n\}_{n=1}^{\infty}$  of positive numbers,  $\lim_{n\to\infty} \alpha_n = 0$  if and only if  $\lim_{n\to\infty} F(\alpha_n) = -\infty$ ;
- (*F*3) There exist  $k \in (0, 1)$  such that  $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$ .

**Definition 1.2.** [9] Let (X,d) be a metric space. A mapping  $T: X \to X$  is said to be an *F*-contraction on (X,d) if there exists  $F \in \mathscr{F}$  and  $\tau > 0$  such that

(1) 
$$\forall x, y \in X, \ [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \le F(d(x, y))],$$

Wardowski [9] stated a modified version of Banach contraction principle as follows:

**Theorem1.2.** [9] Let (X,d) be a complete metric space and let  $T : X \to X$  be an *F*-contraction. Then *T* has a unique fixed point  $x^* \in X$  and for every  $x \in X$  the sequence  $\{T^n x\}_{n \in \mathbb{N}}$  converges to  $x^*$ .

Later, Wardowski and Dung [8] introduced the notion of an F-weak contraction as follows and proved a fixed point theorem for F-weak contractions.

**Definition 1.3.** [8] Let (X,d) be a metric space. A mapping  $T: X \to X$  is said to be an *F*-weak contraction on (X,d) if there exists  $F \in \mathscr{F}$  and  $\tau > 0$  such that

(2) 
$$\forall x, y \in X, \ [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \le F(M(x, y))],$$

where,

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\}.$$

By using the notion of F-weak contraction, Wardowski and Dung [8] proved a fixed point theorem which generalizes the result of Wardowski [9].

**Theorem 1.3.** [8] Let (X,d) be a complete metric space and let  $T : X \to X$  be an *F*-weak contraction. If *T* or *F* is continuous, then *T* has a unique fixed point  $x^* \in X$  and for every  $x \in X$  the sequence  $\{T^n x\}_{n \in \mathbb{N}}$  converges to  $x^*$ .

Recall that, for a map  $T: X \to X$  on a metric space (X, d), contraction conditions usually contained at most five values d(x, y), d(x,T x), d(y,Ty), d(x,Ty), d(y,T x), (see [3, 7] for example). Recently, by adding four new

# FIXED POINT THEOREMS FOR GENERALIZED $F - \varphi$ – WEAK CONTRACTIONS IN COMPLETE METRIC SPACES3 values $d(T^2x, x)$ , $d(T^2x, Tx)$ , $d(T^2x, y)$ , $d(T^2x, Ty)$ to a contraction condition, Kumam et al. [5] stated a new generalization of Ćirić fixed point theorem in [2].

Recently, by adding values  $d(T^2x, x)$ ,  $d(T^2x, Tx)$ ,  $d(T^2x, y)$ ,  $d(T^2x, Ty)$  to (2), Dung and Hang [4] introduced the notion of a generalized F-contraction and proved a fixed point theorem for such maps. They give examples to show that their result is a proper extension of Theorem and some others in the literature.

Dung and Hang [4] generalized an F-weak contraction to a generalized F-contraction as follows: **Definition 1.4.** Let (X,d) be a metric space. A mapping  $T: X \to X$  is said to be a generalized *F*-contraction on (X,d) if there exists  $F \in \mathscr{F}$  and  $\tau > 0$  such that

$$\forall x, y \in X, \ [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \le F(M_T(x, y)),$$

where

$$M_T(x,y) = \left( \max \left\{ \begin{array}{c} d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}, \\ \frac{d(T^2x,x)+d(T^2x,Ty)}{2}, d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty) \end{array} \right\} \right)$$

By using the notion of generalized F- contraction, Dung and Hang have proved the following fixed point theorem which generalizes the result of Wardowski and Dung [8].

**Theorem1.4.** [4] Let (X,d) be a complete metric space and let  $T : X \to X$  be a generalized *F*-contraction. If *T* or *F* is continuous, then *T* has a unique fixed point  $x^* \in X$  and for every  $x \in X$  the sequence  $\{T^n x\}_{n \in \mathbb{N}}$  converges to  $x^*$ .

Very recently, Piri and Kumam [6] described a large class of functions by replacing the condition (F3) in the definition of F-contraction introduced by Wardowski [9] with the following one:

(F3') F is continuous on  $(0, \infty)$ .

Let  $\mathfrak{F}$  denote the family of all functions  $F : \mathbb{R}_+ \to \mathbb{R}$  which satisfy conditions (F1), (F2) and (F3').

**Remark 1.1.** [6] Note that, the conditions (F3) and (F3') are independent of each other. Indeed, for  $p \ge 1$ ,  $F(\alpha) = \frac{-1}{\alpha^p}$  satisfies the conditions (F1) and (F2) but it does not satisfy (F3), while it satisfies the condition (F3'). Therefore,  $\mathfrak{F} \not\subseteq \mathscr{F}$ . Again, for a > 1,  $t \in (0, 1/a)$ ,  $F(\alpha) = \frac{-1}{(\alpha + \lfloor \alpha \rfloor)^r}$ , where  $\lfloor \alpha \rfloor$  denotes the integral part of  $\alpha$ , satisfies the conditions (F1) and (F2) but it does not satisfy (F3'), while it satisfies the condition (F3) for any  $k \in (1/a, 1)$ . Therefore,  $\mathscr{F} \not\subseteq \mathfrak{F}$ . Also, if we take  $F(\alpha) = \ln \alpha$ , then  $F \in \mathscr{F}$  and  $F \in \mathfrak{F}$ . Therefore,  $\mathscr{F} \cap \mathfrak{F} \neq \emptyset$ .

Under this new set-up, they proved some Wardowski and Suzuki type fixed point results in metric spaces as follows:

**Theorem1.5.** [6] Let *T* be a self-mapping of a complete metric space *X* into itself. Suppose, there exist  $F \in \mathfrak{F}$  and  $\tau > 0$  such that

$$\forall x, y \in X, \ [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \le F(d(x, y))].$$

Then, T has a unique fixed point  $x^* \in X$  and for every  $x_0 \in X$  the sequence  $\{T^n x_0\}_{n=1}^{\infty}$  converges to  $x^*$ .

**Theorem 1.6.** [6] Let *T* be a self-mapping of a complete metric space *X* into itself. Suppose, there exist  $F \in \mathfrak{F}$  and  $\tau > 0$  such that

$$\forall x, y \in X, \ \frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow \tau + F(d(Tx, Ty)) \le F(d(x, y))].$$

Then, T has a unique fixed point  $x^* \in X$  and for every  $x_0 \in X$  the sequence  $\{T^n x_0\}_{n=1}^{\infty}$  converges to  $x^*$ .

In view of Remark , it is meaningful to consider the result of Wardowski and Van Dung [8] and Dung and Hang [4] with the mappings  $F \in \mathfrak{F}$  instead  $F \in \mathfrak{F}$ .

In this paper, we introduce the notation of generalized  $F - \varphi$ -weak contraction and prove fixed point theorems for generalized  $F - \varphi$ -weak contraction, which is generalization of Theorem 2.2 of [8] and Theorem 3 of [4], also our theorem gives all consequence of Theorem 2.1 of [6] without assumption (F2) used in it's proof.

## 2. main results

Let (F1') be defined as follows:

(*F*1') F is increasing on  $\mathbb{R}_+$ , i.e. for all  $x, y \in \mathbb{R}_+$  such that  $x < y, F(x) \le F(y)$ .

We use  $\mathfrak{F}_G$  to denote the family of all functions  $F : \mathbb{R}_+ \to \mathbb{R}$  which satisfy conditions (F1') and (F3') and  $\Psi$  to denote the set of all increasing functions  $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ .

**Theorem 2.1.** Let (X,d) be a complete metric space and let  $T, S : X \to X$  be self mappings such that

(3) 
$$\forall x, y \in X, \ d(Tx, Sy) > 0 \Rightarrow F(d(Tx, Sy)) \le F(M_{TS}(x, y)) - \varphi(M_{TS}(x, y)),$$

where  $F \in \mathfrak{F}_G$ ,  $\varphi \in \Psi$  and

$$M_{TS}(x,y) = \max \left\{ \begin{array}{c} d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Sy) + d(Tx,Y)}{2}, \frac{d(Tx,TSy) + d(y,TSy)}{2}, \\ d(Tx,TSy), d(Sy,TSy), d(x,Sy), d(x,TSy), \end{array} \right\}$$

Then there exists a unique point  $x^* \in X$  such that  $x^* = Tx^*$  or  $x^* = Sx^*$  or  $x^* = Tx^* = Sx^*$ .

**Proof.** First, we show that  $M_{TS}(x,y) = 0$  if and only if x = y = Tx = Ty = Sx = Sy. Let  $M_{TS}(x,y) = 0$ . Then from  $d(x,y) \le M_{TS}(x,y)$ ,  $d(x,Tx) \le M_{TS}(x,y)$  and  $d(Sy,y) \le M_{TS}(x,y)$ , we have x = y = Tx = Ty = Sx = Sy. If x = y = Tx = Ty = Sx = Sy, then  $M_{TS}(x,y) = 0$ . Let  $x_0 \in X$ . Putting  $x_1 = Tx_0$ ,  $x_2 = Sx_1$ , then  $x_3 = Tx_2$  and  $x_4 = Sx_3$ . Inductively, choose a sequence  $\{x_n\}_{n \in \mathbb{N}}$  such that

$$x_{2n+1} = Tx_{2n}$$
 and  $x_{2n+2} = Sx_{2n+1}$ ,

for all  $n \in \mathbb{N}_0$ . Suppose that there exists  $n \in \mathbb{N}$  such that  $d(x_n, x_{n+1}) = 0$ . If n is an even, we have  $d(Sx_{n-1}, Tx_n) = d(x_n, x_{n+1}) = 0$ , therefore  $x_n = Sx_{n-1} = Tx_n$  and the proof is complete. If n is an odd, we have  $d(Tx_{n-1}, Sx_n) = d(x_n, x_{n+1}) = 0$ .

(4) 
$$F(d(x_{n+1},x_n)) = F(d(Tx_n,Sx_{n-1})) \le F(M_{TS}(x_n,x_{n-1})) - \varphi(M_{TS}(x_n,x_{n-1})),$$

where

(5)

(7)

$$\begin{split} M_{TS}(x_n, x_{n-1}) &= \max \left\{ \begin{array}{l} d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Sx_{n-1}), \\ \frac{d(x_n, Sx_{n-1}) + d(Tx_n, x_{n-1})}{2}, \\ \frac{d(Tx_n, TSx_{n-1}) + d(x_{n-1}, TSx_{n-1})}{2}, \\ d(x_n, Sx_{n-1}), d(x_n, TSx_{n-1}), \\ d(Tx_n, TSx_{n-1}), d(x_n, TSx_{n-1}), \\ d(Tx_n, TSx_{n-1}), d(Sx_{n-1}, TSx_{n-1}) \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), \\ \frac{d(x_n, x_n) + d(x_{n+1}, x_{n-1})}{2}, \\ \frac{d(x_n, x_n), d(x_n, x_{n+1})}{2}, \\ d(x_n, x_n), d(x_n, x_{n+1}), \\ d(x_{n+1}, x_{n+1}), d(x_n, x_{n+1}), \\ d(x_{n+1}, x_{n+1}), d(x_n, x_{n+1}) \end{array} \right\} \end{split}$$

If there exists  $n \in \mathbb{N}$  such that  $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$  then from (4), (5) and (*F*1'), we get

$$F(d(x_{n+1},x_n)) \le F(d(x_{n+1},x_n)) - \varphi(M_{TS}(x_n,x_{n-1})).$$

Since  $\varphi(M_{TS}(x_n, x_{n-1})) > 0$ , so we get a contradiction. Therefore

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n), \forall n \in \mathbb{N}.$$

Thus, from (4), (5) and (F1'), we have

(6)  

$$F(d(x_{n+1},x_n)) = F(d(Tx_n,Sx_{n-1})) \le F(M_{TS}(x_n,x_{n-1})) - \varphi(M_{TS}(x_n,x_{n-1}))$$

$$\le F(d(x_n,x_{n-1})) - \varphi(d(x_n,x_{n-1}))$$

If n is an odd, we also obtain

$$F(d(x_n, x_{n+1})) = F(d(Tx_{n-1}, Sx_n)) \le F(x_{n-1}, (x_{n-1}, x_n)) - \varphi(M_{TS}(x_{n-1}, x_n))$$
$$\le F(d(x_{n-1}, x_n)) - \varphi(d(x_{n-1}, x_n))$$

It follows from (6) and (7) that

(8)  

$$F(d(x_{n}, x_{n+1})) \leq F(M_{TS}(x_{n-1}, x_{n})) - \varphi(M_{TS}(x_{n-1}, x_{n}))$$

$$\leq F(d(x_{n-1}, x_{n})) - \varphi(d(x_{n-1}, x_{n})), \quad \forall n \in \mathbb{N}.$$

Since  $\varphi(d(x_{n-1}, x_n)) > 0$ , so from (F1'), we give

$$d(x_n, x_{n+1}) \le d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}$$

Therefore  $\{d(x_{n+1},x_n)\}_{n\in\mathbb{N}}$  is a nonnegative nonincreasing sequence of real numbers, and hence

(9) 
$$\lim_{n \to \infty} d(x_{n+1}, x_n) = \gamma \ge 0$$

Now, we claim that  $\gamma = 0$ . On the contrary, we assume that  $\gamma > 0$ . Since  $\{d(x_{n+1}, x_n)\}_{n \in \mathbb{N}}$  is a nonnegative decreasing sequence, so there exists  $N_1 \in \mathbb{N}$  such that

(10) 
$$d(x_{n+1},x_n) > \gamma, \quad \forall n > N_1.$$

Since  $\varphi$  is increasing, so from (8), (10) and (*F*1'), we get

$$F(\gamma) \le F(d(x_{n+1}, x_n)) \le F(d(x_{n-1}, x_n)) - \varphi(d(x_{n-1}, x_n))$$
  
$$\le F(d(x_{n-1}, x_n)) - \varphi(r)$$
  
$$\le F(d(x_{n-2}, x_{n-1})) - \varphi(d(x_{n-2}, x_{n-1})) - \varphi(r)$$
  
$$\le F(d(x_{n-2}, x_{n-1})) - 2\varphi(r)$$
  
$$\vdots$$
  
$$\le F(d(x_{N_1}, x_{N_1+1})) - (n - N_1)\varphi(r), \quad \forall n > N_1.$$

Since  $F(\gamma) \in \mathbb{R}$  and  $\lim_{n \to \infty} [F(d(x_{N_1}, x_{N_1+1})) - (n - N_1)\varphi(r)] = -\infty$ . So there exists  $N_2 \in \mathbb{N}$  such that

(12) 
$$F(d(x_{N_1}, x_{N_1+1})) - (n - N_1)\varphi(r) < F(\gamma), \quad \forall n > N_2.$$

Setting  $N_3 = \max\{N_1, N_2\}$ . From (11) and (12), we get

$$F(\boldsymbol{\gamma}) \leq F(d(\boldsymbol{x}_{N_1}, \boldsymbol{x}_{N_1+1})) - (n - N_1)\boldsymbol{\varphi}(r) < F(\boldsymbol{\gamma}), \quad \forall n > N_3.$$

This contradiction prove that

(11)

(13)

$$\lim_{n\to\infty}d(x_n,x_{n+1})=0.$$

## FIXED POINT THEOREMS FOR GENERALIZED $F - \phi -$ WEAK CONTRACTIONS IN COMPLETE METRIC SPACES7

Now, we claim that,  $\{x_n\}_{n=1}^{\infty}$  is a Cauchy sequence. Let

$$p_n = \sup\{d(x_i, x_j) : i, j \ge n\}, \quad \forall n \in \mathbb{N}.$$

Obviously  $\{p_n\}_{n=1}^{\infty}$  is a nonnegative decreasing sequence of real numbers, and hence

(14) 
$$\lim_{n \to \infty} p_n = p \ge 0.$$

We prove that p = 0. Reasoning contradiction, assume that p > 0. Choose  $\varepsilon < \frac{7}{33}p$  small enough. From (13) and (14) there exists  $N_4 \in \mathbb{N}$  such that

(15) 
$$d(x_n, x_{n+1}) < \frac{5}{28}\varepsilon, \text{ and } p - \varepsilon < p_n < p + \varepsilon, \quad \forall n \ge N_4$$

Since  $p_{N_{4}+1} > p - \varepsilon$ . So there exists  $i, j \in \mathbb{N}$  such that

$$i, j \geq N_4 + 1$$
 and  $d(x_i, x_j) > p - \varepsilon$ .

Since

$$p - \varepsilon < d(x_i, x_j) \le d(x_i, x_{i+1}) + d(x_{i+1}, x_j) < \varepsilon + d(x_{i+1}, x_j)$$

and

$$p-\varepsilon < d(x_i, x_j) \le d(x_j, x_{j+1}) + d(x_{j+1}, x_i) < \varepsilon + d(x_{j+1}, x_i).$$

Therefore

$$p-2\varepsilon < d(x_{i+1},x_j)$$
 and  $p-2\varepsilon < d(x_i,x_{j+1})$ 

So, we can assume that, i is odd, j is even, and  $d(x_i, x_j) > p - 2\varepsilon$ . It follows from (15) that

(16)  
$$p - 2\varepsilon < d(x_i, x_j) \le d(x_i, x_{i-1}) + d(x_{i-1}, x_{j-1}) + d(x_{j-1}, x_j) < \frac{10}{28}\varepsilon + d(x_{i-1}, x_{j-1}).$$

Therefore,

(17)  
$$M_{TS}(x_{i-1}, x_{j-1}) \ge d(x_{i-1}, x_{j-1}) > p - 2\varepsilon - \frac{10}{28}\varepsilon$$
$$= p - \frac{66}{28}\varepsilon > p - \frac{33}{14} \times \frac{7}{33}p = \frac{1}{2}p.$$

Using the fact that  $\varphi$  is increasing with respect to (8), (16), (17) and (F1') we deduce that

(18)  

$$F(p-\varepsilon) \leq F(d(x_{i},x_{j})) = F(d(Tx_{i-1},Sx_{j-1}))$$

$$\leq F(M_{TS}(x_{i-1},x_{j-1})) - \varphi(M_{TS}(x_{i-1},x_{j-1}))$$

$$\leq F(M_{TS}(x_{i-1},x_{j-1})) - \varphi(\frac{1}{2}p),$$

where

$$\begin{split} M_{TS}(x_{i-1},x_{j-1}) &= \max \begin{cases} d(x_{i-1},x_{j-1}), d(x_{i-1},Tx_{i-1}), d(x_{j-1},Sx_{j-1}), \\ \frac{d(x_{i-1},Sx_{j-1})+d(x_{j-1},TSx_{j-1})}{2}, \\ \frac{d(x_{i-1},Sx_{j-1}), d(x_{i-1},TSx_{j-1}), \\ d(Tx_{i-1},TSx_{j-1}), d(x_{j-1},TSx_{j-1}), \\ d(Tx_{i-1},TSx_{j-1}), d(Sx_{j-1},TSx_{j-1}) \end{cases} \\ &= \max \begin{cases} d(x_{i-1},x_{j-1}), d(x_{i-1},x_{i}), d(x_{j-1},x_{j}), \\ \frac{d(x_{i-1},x_{j})+d(x_{i},x_{j-1})}{2}, \\ \frac{d(x_{i-1},x_{j}), d(x_{i-1},x_{j+1}), \\ d(x_{i-1},x_{j}), d(x_{i-1},x_{i}), d(x_{j-1},x_{j}), \\ \frac{d(x_{i-1},x_{j-1}), d(x_{i-1},x_{i+1}), \\ d(x_{i-1},x_{j-1}), d(x_{i-1},x_{i-1}) + d(x_{j-1},x_{j-1}), \\ \frac{d(x_{i-1},x_{j-1})+d(x_{j-1},x_{j})+d(x_{j},x_{j+1}), \\ \frac{d(x_{i-1},x_{j-1})+d(x_{j-1},x_{j})+d(x_{j},x_{j+1}), \\ \frac{d(x_{i-1},x_{j-1})+d(x_{j-1},x_{j})+d(x_{j},x_{j+1}), \\ d(x_{i},x_{i-1})+d(x_{i-1},x_{j-1})+d(x_{j-1},x_{j})+d(x_{j},x_{j+1}), \\ d(x_{i-1},x_{j-1})+d(x_{i-1},x_{j-1})+d(x_{j-1},x_{j})+d(x_{j},x_{j+1}), \\ d(x_{i-1},x_{j-1})+d(x_{i-1},x_{j-1})+\frac{5}{28}\varepsilon, \\ \frac{1}{2}d(x_{i-1},x_{j-1})+\frac{10}{28}\varepsilon, d(x_{i-1},x_{j-1})+\frac{5}{28}\varepsilon, \frac{5}{28}\varepsilon \end{cases} \\ &\leq \max \begin{cases} d(x_{i-1},x_{j-1}), \frac{10}{28}\varepsilon, d(x_{i-1},x_{j-1})+\frac{5}{28}\varepsilon, \frac{5}{28}\varepsilon \end{cases} \\ \\ \leq \max \begin{cases} d(x_{i-1},x_{j-1})+\varepsilon \\ \leq p_{N_4}+\varepsilon < p+2\varepsilon. \end{cases} \end{cases}$$

It follows from (18), (19) and (F1') that

(19)

(20) 
$$F(p-2\varepsilon) \le F(p+2\varepsilon) - \varphi(\frac{1}{2}p).$$

From (F3), as  $\varepsilon \to 0$ , we get  $F(p) \le F(p) - \varphi(\frac{1}{2}p)$ . Since  $\varphi(\frac{1}{2}p) > 0$ , this is impossible. Thus, we must have p = 0. That is, the sequence  $\{x_n\}_{n=1}^{\infty}$  is a Cauchy sequence. By completeness of (X,d),  $\{x_n\}_{n=1}^{\infty}$  converges to some point  $x^*$  in X. Therefore,

(21) 
$$\lim_{n \to \infty} d(x_{2n}, x^*) = \lim_{n \to \infty} d(x_{2n+1}, x^*) = \lim_{n \to \infty} d(x_n, x^*) = 0.$$

Now, we will show that

$$x^* = Tx^* \quad and \quad x^* = Sx^*.$$

Arguing by contradiction, we assume that

$$d(x^*, Tx^*) > 0$$
 or  $d(x^*, Sx^*) > 0$ .

If  $d(x^*, Tx^*) > 0$ . Using the fact that  $\varphi$  is increasing with respect to (3), we deduce that

(22)  

$$F(d(Tx^*, x_{2n+2})) = F(d(Tx^*, Sx_{2n+1})) \le F(M_{TS}(x^*, x_{2n+1})) - \varphi(M_{TS}(x^*, x_{2n+1}))$$

$$\le F(M_{TS}(x^*, x_{2n+1})) - \varphi(x^*(x^*, Tx^*)),$$

where,

$$\begin{split} d(x^*,Tx^*) &\leq M_{TS}(x^*,x_{2n+1}) \\ &= \max \left\{ \begin{array}{l} d(x^*,x_{2n+1}), d(x^*,Tx^*), d(x_{2n+1},Sx_{2n+1}), \\ \frac{d(x^*,Sx_{2n+1})+d(Tx^*,x_{2n+1})}{2}, \\ \frac{d(Tx^*,TSx_{2n+1})+d(x_{2n+1},TSx_{2n+1})}{2}, \\ d(Tx^*,TSx_{2n+1}), d(Sx_{2n+1},TSx_{2n+1}), \\ d(x^*,Sx_{2n+1}), d(x^*,Tx^*), d(x_{2n+1},x_{2n+2}), \\ \frac{d(x^*,x_{2n+2})+d(Tx^*,x_{2n+1})}{2}, \\ \frac{d(Tx^*,x_{2n+3})+d(x_{2n+1},x_{2n+3})}{2}, \\ d(Tx^*,x_{2n+3}), d(x_{2n+2},x_{2n+3}), \\ d(x^*,x_{2n+2}), d(x^*,Tx^*), d(x_{2n+1},x_{2n+2}), \\ \frac{d(x^*,x_{2n+2})+d(Tx^*,x^*)+d(x^*,x_{2n+3}), \\ d(x^*,x_{2n+2}), d(x^*,x_{2n+3}), \\ d(x^*,x_{2n+3}), d(x_{2n+1},x_{2n+3}), \\ d(Tx^*,x^*)+d(x^*,x_{2n+3}), \\ d(Tx^*,x^*)+d(x^*,x_{2n+3}), \\ d(Tx^*,x^*)+d(x^*,x_{2n+3}), \\ d(x_{2n+2},x_{2n+3}), d(x^*,x_{2n+2}), d(x^*,x_{2n+3}), \\ \end{array} \right\} \\ &= d(x^*,Tx^*). \end{split}$$

So, we get

$$\lim_{n \to \infty} M_{TS}(x^*, x_{2n+1}) = d(x^*, Tx^*).$$

(23)

From (22), (23), (F3') and letting  $n \rightarrow \infty$ , we have

$$F(d(x^*, Tx^*)) \le F(d(x^*, Tx^*)) - \varphi(d(x^*, Tx^*)).$$

Since  $\varphi(d(x^*, Tx^*)) > 0$ , we obtain a contradiction. If  $d(x^*, Sx^*) > 0$ . By using an argument similar to the above, we obtain a contradiction. Hence

$$x^* = Tx^* = Sx^*.$$

If there exists another point  $y^* \in X$  such that  $y^* = Ty^* = Sy^*$ , then using an argument similar to the above, we get

$$\begin{split} F(d(x^*, y^*)) &= F(d(Tx^*, Ty^*)) \leq F(M_{TS}(x^*, y^*)) - \varphi(M_{TS}(x^*, y^*)) \\ &= F(d(x^*, y^*)) - \varphi(d(x^*, y^*)) \\ &< F(d(x^*, y^*)). \end{split}$$

Hence  $x^* = y^*$ . The proof is completed.

**Theorem 2.2.** Let (X,d) be a complete metric space and  $T, S: X \to X$  two mappings such that

$$\forall x, y \in X, \ d(Tx, Sy) > 0 \Rightarrow F(d(Tx, Sy)) \le F(M_{TS}(x, y)) - \varphi(d(x, y)),$$

where  $F \in \mathfrak{F}_G$ ,  $\varphi \in \Psi$  and

$$M_{TS}(x,y) = \max \left\{ \begin{array}{c} d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Sy) + d(Tx,y)}{2}, \frac{d(Tx,TSy) + d(y,TSy)}{2}, \\ d(Tx,TSy), d(Sy,TSy), d(x,Sy), d(x,TSy), \end{array} \right\}.$$

Then there exists a unique point  $x^* \in X$  such that  $x^* = Tx^*$  or  $x^* = Sx^*$  or  $x^* = Tx^* = Sx^*$ .

**Theorem 2.3.** Let (X,d) be a complete metric space and  $T, S: X \to X$  two mappings such that

$$\forall x, y \in X, \ d(Tx, Sy) > 0 \Rightarrow F(d(Tx, Sy)) \le F(d(x, y)) - \varphi(d(x, y)),$$

where  $F \in \mathfrak{F}_G$  and  $\varphi \in \Psi$ . Then there exists a unique point  $x^* \in X$  such that  $x^* = Tx^*$  or  $x^* = Sx^*$  or  $x^* = Tx^* = Sx^*$ .

**Theorem 2.4.** Let (X,d) be a complete metric space and  $T, S : X \to X$  two mappings such that for all  $x, y \in X$ ,

(24) 
$$d(Tx,Sy) \le M_{TS}(x,y) - \varphi(M_{TS}(x,y)),$$

where  $\varphi : [0,\infty) \to [0,\infty)$  is a increasing function with  $\varphi(t) > 0$  for  $t \in (0,\infty)$ ,  $\varphi(0) = 0$  and

$$M_{TS}(x,y) = \max \left\{ \begin{array}{c} d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Sy) + d(Tx,y)}{2}, \frac{d(Tx,TSy) + d(y,TSy)}{2}, \\ d(Tx,TSy), d(Sy,TSy), d(x,Sy), d(x,TSy), \end{array} \right\}.$$

Then there exists a unique point  $x^* \in X$  such that  $x^* = Tx^*$  or  $x^* = Sx^*$  or  $x^* = Tx^* = Sx^*$ .

**Proof.** It suffices to take F = I in Theorem 2.1.

# FIXED POINT THEOREMS FOR GENERALIZED $F - \varphi -$ WEAK CONTRACTIONS IN COMPLETE METRIC SPACES **Theorem 2.5.** Let (X, d) be a complete metric space and $T, S : X \to X$ two mappings such that for all $x, y \in X$ ,

(25) 
$$d(Tx,Ty) \le d(x,y) - \varphi(d(x,y)),$$

where  $\varphi : [0,\infty) \to [0,\infty)$  is a increasing function with  $\varphi(t) > 0$  for  $t \in (0,\infty)$  and  $\varphi(0) = 0$ . Then there exists a unique point  $x^* \in X$  such that  $x^* = Tx^*$ .

**Proof.** It suffices to take T=S and F = I in Theorem 2.3.

By the careful analysis of the proof of Theorem , we have the following theorems. Because its proof is much simpler than that of Theorem , we omit Their proof.

**Theorem2.6.** Let (X,d) be a complete metric space and let  $T : X \to X$  be a generalized *F*- contraction. If *F* is continuous, then *T* has a unique fixed point  $x^* \in X$  and for every  $x \in X$  the sequence  $\{T^n x\}_{n \in \mathbb{N}}$  converges to  $x^*$ .

**Proof.** It suffices to take T = S and  $\varphi = \tau$  in Theorem 2.2.

**Theorem2.7.** Let (X,d) be a complete metric space and let T be a self mapping on X. If there exists  $F \in \mathscr{F}_G$  and  $\tau > 0$  such that

$$\forall x, y \in X, \ [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \le F(d(x, y))].$$

Then T has a unique fixed point  $x^* \in X$  and for every  $x \in X$  the sequence  $\{T^n x\}_{n \in \mathbb{N}}$  converges to  $x^*$ .

**Proof.** It is suffices to take  $\varphi = \tau$  in Theorem 2.3.

**Remark 2.1.** Our theorems improve and extend the following theorems in the following aspects.

- (1) Since F is strictly increasing, Theorem 2.2 gives all consequence of Theorem 2.1 of [10] by assumption " $\varphi$  is increasing" instead of the assumption " $\varphi$  is lower semi-continuous".
- (2) Since F is strictly increasing, by taking  $\varphi = 0$  in Theorem 2.2, Theorem 2.2 gives all consequence of Theorem 2.1 of [8] without assumptions (F2) and (F3) used in it's proof.
- (3) Theorem 2.5 gives all consequence of Theorem 1 of [7] without assumption " continuity of the function φ" used in it's proof.
- (4) If in Theorem 3 of [4] F be continuouse, Theorem 2.6 gives all consequence of Theorem 3 of [4] without assumptions (F2) and (F3) used in it's proof.
- (5) Theorem 2.7 gives all consequence of Theorem 2.1 of [6] without assumption (F2) used in it's proof.

## REFERENCES

 I. Alber, Ya and S. Guerre-Delabriere, Principles of weakly contractive maps in Hilbert spaces, New Results in Operator Theory, Advances and Appl., 98, (I. Gohberg and Yu. Lyubich, eds.), Birkhauser, Basel, pp.(1997), 7-22.

- [2] L.B. Ćirić: A generalization of Banachs contraction principle, Proc. Am. Math. Soc. 45 (1974), 267-273.
- [3] P. Collaço and J.C. Silva: A complete comparison of 25 contraction conditions. Nonlinear Anal. 30 (1997), 471-476.
- [4] N.V. Dung and V.L. Hang: A fixed point theorem for generalized F-contractions on complete metric spaces, Vietnam J. Math. in press.
- [5] P. Kumam, N.V. Dung and K. Sitthithakerngkiet: A generalization of Ćirić fixed point theorem, Filomat (2014). Accepted
- [6] H. Piri and P. Kumam, Some fixed point theorems concerning F-contraction in complete metric spaces, Fixed Point Theory and Applications 2014 (2014), Article ID 210.
- [7] B.E. Rhoades: A comparison of various definitions of contractive mappings. Trans. Amer. Math. Soc. 226 (1977), 257-290.
- [8] D. Wardowski and N. Van Dung, Fixed points of f-weak contractions on complete metric spaces, Demonstratio Math. 1 (2014), 146-155.
- [9] D. Wardowski, Fixed point theory of a new type of contractive mappings in complete metric spaces, Fixed PoinTheory Appl. 2012 (2012), Article ID 94.
- [10] Q. Zhang, Y. Song, fixed point theory for generalized  $\varphi$ -weak contractions, Appl. Math. Lett. 22 (2009), 75-78.