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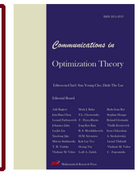

CONVERGENCE ANALYSIS OF A HALPERN-TYPE ITERATIVE ALGORITHM FOR ZERO POINTS OF ACCRETIVE OPERATORS

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Abstract. In this paper, we study zero points of accretive operators based on a Halpern-type iterative algorithm. Strong convergence of the iterative algorithm is obtained in the framework of reflexive Banach spaces. We also apply our main results to solve minimizer problems of a convex function.

Keywords. Accretive operator; Convex optimization; Fixed point; Monotone operator; Viscosity approximation.

1. Introduction-Preliminaries

Let E be a real Banach space and let E^* be the dual space of E. Let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E.$$

Let $B_E = \{x \in E : \|x\| = 1\}$. The modulus of convexity of E is defined by

$$\delta(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon\}$$

for every ε with $0 \le \varepsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then

$$\left\|\frac{x+y}{2}\right\| \le r\Big(1 - \delta(\frac{\varepsilon}{r})\Big)$$

for every $x, y \in E$ with $\|x\| \le r$, $\|y\| \le r$ and $\|x-y\| \ge \varepsilon$.

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E is said to be smooth or said to be have a Gâteaux differentiable norm if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$$

exists for each $x,y\in B_E$. E is said to have a uniformly Gâteaux differentiable norm if for each $y\in B_E$, the limit is attained uniformly for all $x\in B_E$. E is said to be uniformly smooth or said to be have a uniformly Fréchet differentiable norm if the limit is attained uniformly for $x,y\in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single valued and uniformly norm to weak* continuous on each bounded subset of E.

Recall that a mapping $T:C\to C$ is said to be α-contractive iff there exists a constant $\alpha\in(0,1)$ such that if

$$\|Tx-Ty\|\le\alpha\|x-y\|,\quad \forall x,y\in C.$$

T is said to be nonexpanisve if

$$\|Tx-Ty\|\le\|x-y\|,\quad \forall x,y\in C.$$

In this paper, we use $F(T)$ to denote the set of fixed points of T. It is known that the fixed point set of nonexpansive mappings is not empty provided that C is closed convex bounded and E is uniformly convex. A closed convex subset C of E is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C into itself has a fixed point in D.

Let D be a nonempty subset of set C. A mapping $Q_D:C\to D$ is said to be a contraction if $Q_D^2=Q_D$. It is called sunny if for each $x\in C$ and $t\in(0,1)$, we have $Q_Dx=Q_D\big(tx+(1-t)Q_Dx\big)$. Q_D is said to be a sunny nonexpansive retractction if Q_D is sunny, nonexpansive and a contraction. D is said to be a nonexpansive retract of C if there exists a nonexpansive retraction from C onto D.

The following result, which was established in [1] nd [2], describes a characterization of sunny nonexpansive retractions in the framework of real smooth Banach spaces.

Let E be a real smooth Banach space and C be a nonempty subset of E. Let $Q_C:E\to C$ be a retraction and $\mathfrak{J}$ be the normalized duality mapping on E. Then the following are equivalent:

(a) Q_C is nonexpansive and sunny;

(b) $\langle x-Q_Cx,\mathfrak{J}(y-Q_Cx)\rangle\le 0, \forall x\in E, y\in C.$

It is well known that if E is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection from E onto C. Let C be a nonempty convex closed subset of a real smooth Banach space E, $x_0 \in C$ and $x \in E$. Then $Q_C x = x_0$ if and only if $0 \geq \langle x - x_0, \mathfrak{J}(y - x_0) \rangle$ for all $y \in C$, where Q_C is a sunny nonexpansive retraction from E onto C.

In the sequel, we use j to denote the single-valued normalized duality mapping. Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$ is said to be accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that

$$\langle y_1 - y_2, j(x_1 - x_2) \rangle \geq 0.$$

An accretive operator A is said to satisfy the range condition if

$$\overline{D(A)} \subset \cap_{r>0} R(I + rA),$$

where $\overline{D(A)}$ denote the closure of $D(A)$. An accretive operator A is said to be m-accretive if $R(I + rA) = E$ for all $r > 0$. In a real Hilbert space, an operator A is m-accretive if and only if A is maximal monotone.

For an accretive operator A, we can define a nonexpansive single-valued mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$ for each $r > 0$, which is called the resolvent of A. We also define the Yosida approximation A_r by $A_r = \frac{1}{r}(I - J_r)$. It is known that $A_r x \in AJ_r x$ for all $x \in R(I + rA)$ and $\|A_r x\| \leq \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$.

For finding a zero point of accretive operators, a powerful and successful algorithm was introduced by Rockafellar [3] which is recognized as the Rockafellar's proximal point algorithm: for any initial point $x_0 \in E$, a sequence $\{x_n\}$ is generated by

$$x_{n+1} = J^A_{r_n}(x_n + e_n), \quad \forall n \geq 0,$$

where $J_{r_n} = (I + r_n A)^{-1}$ is the resolvent of A. Moreover, Rockafellar also proved the weak convergence of sequence $\{x_n\}$ when regularization sequence $\{r_n\}$ remains bounded away from zero and error sequence $\{e_n\}$ with restriction $\sum_{n=1}^{\infty} \|e_n\| < \infty$. To find the strong convergence, Bruck [4] proposed the following algorithm: for any initial point $x_0 \in E$ and fixed point $u \in E$,

$$x_{n+1} = J_{r_n} u, \quad \forall n \geq 0.$$

He obtained a strong convergence result on zero points of accretive operators.

The convergence of the proximal point algorithm has been studied by many authors; see, for example, [5]-[17] and the references therein. In this paper, motivated by the research work going on in this direction, we introduce and analysis Halpern-type iterative algorithms with errors. Strong convergence theorems are established in a real Banach space.

2. Lemmas

In this section, we provide some lemmas which play an important role in this article.

Lemma 2.1. [15] *Let E be a real reflexive Banach space whose norm is uniformly Gâteaux differentiable and $A \subset E \times E$ be an accretive operator. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty, closed and convex subset of E such that $\overline{D(A)} \subset C \subset \cap_{t>0} R(I + tA)$. If $A^{-1}(0) \neq \emptyset$, then the strong limit $\lim_{t\to\infty} J_t x$ exists and belongs to $A^{-1}(0)$ for all $x \in C$, where $J_t = (I + tA)^{-1}$ is the resolvent of A for all $t > 0$.*

Lemma 2.2. [18] *Let C be a nonempty closed and convex subset of a uniformly convex Banach space E and $T : C \to C$ a nonexpansive mapping. If a sequence $\{x_n\}$ in C converges weakly to $z \in C$ and $\{x_n - Tx_n\}$ converges strongly to 0 as $n \to \infty$, then $Tz = z$.*

Lemma 2.3. [19] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad n \geq 0,$$

where $\{t_n\}$ is a sequence in $[0, 1]$. Assume that the following conditions are satisfied

(a) $\sum_{n=0}^{\infty} t_n = \infty$ *and* $b_n = o(t_n)$;

(b) $\sum_{n=0}^{\infty} c_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4. [6] In a Banach space E, there holds the inequality

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad x,y \in E,$$

where $j(x+y) \in J(x+y)$.

3. Main results

Theorem 3.1. *Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm and C a nonempty closed and convex subset of E. Let $T : C \to C$ be an α-contractive mapping and $A \subset E \times E$ be an accretive operator with $A^{-1}(0) \neq \emptyset$. Assume that $\overline{D(A)} \subset C \subset \cap_{r>0} R(I+rA)$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E, \\ y_n = (I+r_nA)^{-1}(x_n + e_{n+1}), \\ x_{n+1} = \alpha_n T x_n + \beta_n y_n + \gamma_n f_n, \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$, $\{f_n\} \subset C$ is a bounded sequence, $\{e_n\}$ is a sequence in E, $\{r_n\} \subset (0,\infty)$. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Assume that the following conditions are satisfied

(a) $\alpha_n + \beta_n + \gamma_n = 1$;
(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \|e_n\| < \infty$;
(d) $r_n \to \infty$ *as* $n \to \infty$.

Then the sequence $\{x_n\}$ converges strongly to a zero of A.

Proof. Setting $J_{r_n} = (I+r_nA)^{-1}$ and fixing $p \in A^{-1}(0)$, we find that

$$\begin{aligned} \|y_n - p\| &\leq \|J_{r_n}(x_n + e_{n+1}) - J_{r_n}p\| \\ &\leq \|x_n - p\| + \|e_{n+1}\|. \end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n Tx_n+\beta_n y_n+\gamma_n f_n-p\|\\ &\le \alpha_n\|Tx_n-p\|+\beta_n\|y_n-p\|+\gamma_n\|f_n-p\|\\ &\le \alpha_n\|Tp-p\|+\alpha_n\alpha\|x_n-p\|+\beta_n\|y_n-p\|+\gamma_n\|f_n-p\|\\ &\le \alpha_n\|Tp-p\|+\big(1-\alpha_n(1-\alpha)\big)\|x_n-p\|+\|e_{n+1}\|+\gamma_n\|f_n-p\|.\end{aligned}$$

From the condition (c), we see that the sequence $\{x_n\}$ is bounded, so is, $\{y_n\}$.

Put $w_{n+1}=\alpha_n u+\beta_n y_n+\gamma_n f_n$, where u is a fixed element in C. By Lemma 2.1, we show $\limsup_{n\to\infty}\langle u-z, J(w_{n+1}-z)\rangle\le 0$, where $z=\lim_{t\to\infty}J_t u$. From $A_{r_n}x_n\in AJ_{r_n}x_n$ and $\frac{u-J_tu}{t}\in AJ_tu$, we have $\langle A_{r_n}x_n-\frac{u-J_tu}{t}, J(J_{r_n}x_n-J_tu)\rangle\ge 0$. This implies that

$$\langle tA_{r_n}x_n, J(J_{r_n}x_n-J_tu)\rangle\ge\langle u-J_tu, J(J_{r_n}x_n-J_tu)\rangle.$$

On the other hand, we have

$$\lim_{n\to\infty}\frac{\|x_n-J_{r_n}x_n\|}{r_n}=\lim_{n\to\infty}\|A_{r_n}x_n\|=0.$$

Hence, we have

$$\limsup_{n\to\infty}\langle u-J_tu, J(J_{r_n}x_n-J_tu)\rangle\le 0,\quad \forall t\ge 0.$$

For any $\varepsilon>0$, there exists $t_0>0$ such that

$$\frac{\varepsilon}{2}\ge|\langle J(J_{r_n}x_n-J_tu), z-J_tu\rangle|,$$

and

$$\frac{\varepsilon}{2}\ge|\langle J(J_{r_n}x_n-J_tu)-J(J_{r_n}x_n-z), u-z\rangle|$$

for all $n\ge 0$ and $t\ge t_0$. Hence, we have

$$\begin{aligned}&|\langle u-J_tu, J(J_{r_n}x_n-J_tu)\rangle-\langle u-z, J(J_{r_n}x_n-z)\rangle|\\ &\le|\langle u-z, J(J_{r_n}x_n-J_tu)\rangle-\langle u-z, J(J_{r_n}x_n-z)\rangle|\\ &\quad+|\langle u-J_tu, J(J_{r_n}x_n-J_tu)\rangle-\langle u-z, J(J_{r_n}x_n-J_tu)\rangle|\\ &\le\varepsilon\end{aligned}$$

for all $t\ge t_0$ and $n\ge 0$. Since

$$\varepsilon\ge\limsup_{n\to\infty}\langle u-J_tu, J(J_{r_n}x_n-J_tu)\rangle+\varepsilon\ge\limsup_{n\to\infty}\langle u-z, J(J_{r_n}x_n-z)\rangle,$$

we have $\limsup_{n\to\infty}\langle J(J_{r_n}x_n - z), u - z\rangle \le 0$. On the other hand, we have $\lim_{n\to\infty}\|J_{r_n}x_n - J_{r_n}(x_n + e_{n+1})\| = 0$. Therefore, we have

$$\limsup_{n\to\infty}\langle u - z, J(J_{r_n}(x_n + e_{n+1}) - z)\rangle \le 0. \tag{2.6}$$

Note that

$$\|w_{n+1} - J_{r_n}(x_n + e_{n+1})\| \le \alpha_n\|u - J_{r_n}(x_n + e_{n+1})\| + \gamma_n\|f_n - J_{r_n}(x_n + e_{n+1})\|.$$

This yields that

$$\limsup_{n\to\infty}\langle u - z, J(w_{n+1} - z)\rangle \le 0.$$

On the other hand, we have

$$\begin{aligned}
&\|w_{n+1} - z\|^2\\
&\le (1-\alpha_n)\|(x_n + e_{n+1}) - z\|^2 + 2\alpha_n\langle u - z, J(x_{n+1} - z)\rangle\\
&\quad + 2\gamma_n\|f_n - J_{r_n}(x_n + e_{n+1})\|\|x_{n+1} - z\|\\
&\le (1-\alpha_n)(\|x_n - z\|^2 - 2\langle e_{n+1}, J[(x_n + e_{n+1}) - z]\rangle) + 2\alpha_n\langle u - z, J(x_{n+1} - z)\rangle\\
&\quad + 2\gamma_n\|f_n - J_{r_n}(x_n + e_{n+1})\|\|x_{n+1} - z\|\\
&\le (1-\alpha_n)(\|x_n - z\|^2 + 2\|e_{n+1}\|\|(x_n + e_{n+1}) - z\|) + 2\alpha_n\langle u - z, J(x_{n+1} - z)\rangle\\
&\quad + 2\gamma_n\|f_n - J_{r_n}(x_n + e_{n+1})\|\|x_{n+1} - z\|\\
&\le (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, J(x_{n+1} - z)\rangle\\
&\quad + 2\gamma_n\|f_n - J_{r_n}(x_n + e_{n+1})\|\|x_{n+1} - z\| + 2\|e_{n+1}\|\|(x_n + e_{n+1}) - z\|\\
&\le (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, J(x_{n+1} - z)\rangle + (\gamma_n + \|e_{n+1}\|)B,
\end{aligned}$$

where B is an appropriate constant such that

$$B \ge \max\{\sup_{n\ge 0}\{2\|f_n - J_{r_n}(x_n + e_{n+1})\|\|w_{n+1} - z\|\}, \sup_{n\ge 0}\{2\|(x_n + e_{n+1}) - z\|\}\}$$

It is not hard to see that $\lim_{n\to\infty}\max\{\langle u - z, J(x_{n+1} - z)\rangle, 0\} = 0$. From Lemma 2.3, we have $w_n \to z$. From Suzuki's method [20], we show that $\lim_{n\to\infty}\|x_n - z_n\| = 0$. This proves that $x_n \to z$ as $n \to \infty$. This completes the proof.

In a real Hilbert space, Theorem 3.1 is reduced to the following.

Corollary 3.2. *Let H be a real Hilbert space and C a nonempty, closed and convex subset of H. Let $T : C \to C$ be a contractive mapping and $A \subset H \times H$ a monotone operator with $A^{-1}(0) \ne \emptyset$.*

Assume that $\overline{D(A)} \subset C \subset \cap_{r>0} R(I+rA)$. *Let* $\{x_n\}$ *be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in H, \\ y_n = (I + r_n A)^{-1}(x_n + e_{n+1}), \\ x_{n+1} = \alpha_n T x_n + \beta_n y_n + \gamma_n f_n, \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$, $\{f_n\} \subset C$ *is a bounded sequence,* $\{e_n\}$ *is a sequence in* H, $\{r_n\} \subset (0,\infty)$ *and* $J_{r_n} = (I + r_n A)^{-1}$. *Assume that the following conditions are satisfied*

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(c) $\sum_{n=0}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \|e_n\| < \infty$;

(d) $r_n \to \infty$ *as* $n \to \infty$.

Then the sequence $\{x_n\}$ *converges strongly to a zero of* A.

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