Commun. Optim. Theory 2016 (2016), Article ID 15

MATHRES
Communications in Optimization Theory
Available online at http://cot.mathres.org

COMMON FIXED POINTS OF A FINITE FAMILY OF MULTI-VALUED ρ-NONEXPANSIVE MAPPINGS IN MODULAR FUNCTION SPACES

HABTU ZEGEYE[1,*], WONDIMU W. KASSU[2], MENGISTU G. SANGAGO[2]

[1]Department of Mathematics, Botswana International University of Science and Technology, Private Bag 16, Palapye, Botswana

[2]Department of Mathematics, Addis Ababa Universiry, P.O. Box 31167, Addis Ababa, Ethiopia

Abstract. A Mann-type iterative algorithm is constructed and the sequence of generated in the algorithm is proved to be a common ρ-approximate fixed point sequence for a finite family of multi-valued mappings. Under appropriate conditions on the mappings, the sequence is proved to be ρ-convergent in the framework of modular function spaces.

Keywords. Common fixed point; ρ-nonexapansive mapping; Modular function space; Nonexpansive mapping.

1. Introduction

Modular function spaces are the natural generalizations of both function and sequence variants of many important, from applications perspective, spaces like Lebesgue, Orlicz, Musielak-Orlicz [15], Lorentz, Orlicz-Lorentz, Calderon-Lozanovskii spaces and many others.

The importance of applications of modular function spaces consists of the structure of modular function spaces and are equipped with almost everywhere convergence and convergence in sub-measure. In applications to integral operators, approximation and fixed point results, modular type conditions which are much more natural and modular type assumptions can be more

*Corresponding author.

E-mail addresses: habtuzh@yahoo.com (H. Zegeye), yadikassu@gmail.com (W.W. Kassu), mengistu.goa@aau.edu.et (M.G. Sangago)

Received May 17, 2016

easily verified than their metric or norm counterparts. There are also important results that can be proved only using the apparatus of modular function spaces.

In 1990, Khamsi *et al.* [9] gave an example of a mapping which is ρ-nonexpansive but it is not norm nonexpansive. They demonstrated that for a mapping T to be norm nonexpansive in a modular function space L_ρ, a stronger than $\rho-$nonexpansiveness assumption is needed:

$$\rho(\lambda(Tx-Ty)) \leq \rho(\lambda(x-y)), \quad \forall \lambda \geq 0.$$

From this perspective, the fixed point theory in modular function spaces should be considered as the complementary to the fixed point theory in metric spaces.

The study of this theory in the context of modular function spaces was initiated by Khamsi *et al.* [9] in 1990. Further development of the theory of fixed points in modular function spaces can be found in the exhaustive references of the survey articles [11-13] and the references therein. In fact, most of the work done on fixed points in these spaces was of existential nature.

In 2012, Dehaish and Kozlowski [2] initiated the approximation of fixed points in modular function spaces by Mann iterative process for asymptotically pointwise nonexpansive mappings. In 2014, Abdou *et al.* [1] initiated approximation of common fixed points of two ρ-nonexpansive mappings in modular function space by using the Ishikawa iteration process. However, all the above works have been done for single-valued mappings.

The existence and approximation of fixed points using the well known iteration schemes of Mann [14] and Ishikawa [3] of multi-valued mappings attracted the interests of many mathematicians such as Kozolowski, Latif, Kutib, Abbas etc. Such interest perhaps may be due to its applicability in real world problems, such as Game Theory, Market Economy etc. and other applied mathematics such as nonlinear optimization and differential equations. The Generalization of the existence and approximation of fixed points of multivalued mappings is a natural question once the counterparts of the single valued is known. In 2009, Kutib and Latif [13] worked on the existence of fixed points of multi-valued ρ-nonexpansive mappings in modular function spaces. In 2016, Kassu *et al.* [4] were able to approximate common fixed points of two multi-valued mappings in modular function spaces using the Ishikawa iteration.

In 2014, Khan and Abbas [6] initiated the approximation of fixed points of a multivalued ρ-nonexpansive mapping in modular function spaces by using the Mann iteration process. To be more clear, they proved the following theorems.

Theorem 1.1. [6] *Let ρ satisfy (UUC1) and C be a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : C \to P_\rho(C)$ be a multivalued mapping such that P_ρ^T is a ρ-nonexpansive mapping. Suppose $F_\rho(T) \neq \emptyset$. Let $\{f_n\} \subset C$ be defined by the Mann iterative process*

$$f_{n+1} = (1-\alpha_n)f_n + \alpha_n u_n,$$

where $u_n \in P_\rho^T(f_n)$ and $\{\alpha_n\} \subset (0,1)$ is bounded away from both 0 and 1. Then,

$$\lim_{n\to\infty} \rho(f_n - c) \text{ exists for all } c \in F_\rho(T).$$

and

$$\lim_{n\to\infty} d_\rho(f_n, P_\rho^T(f_n)) = 0.$$

Theorem 1.2. [6] *Let ρ satisfy (UUC1) and C a nonempty ρ-compact, ρ-bounded and convex subset of L_ρ. Let $T : C \to P_\rho(C)$ be a multivalued mapping such that P_ρ^T is ρ-nonexpansive mapping. Suppose that $F_\rho(T) \neq \emptyset$. Let $\{f_n\}$ be defined as in the Theorem 1.1. Then $\{f_n\}$ ρ-converges to a fixed point of T.*

Theorem 1.3. [6] *Let ρ satisfy (UUC1) and C a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : C \to P_\rho(C)$ be a multivalued mapping with $F_\rho(T) \neq \emptyset$ and satisfying condition(I) such that P_ρ^T is ρ-nonexpansive mapping. Let $\{f_n\}$ be as defined in Theorem 1.1. Then, $\{f_n\}$ ρ-converges to a fixed point of T.*

It is our purpose in this paper to construct a Mann-type algorithm and show that the sequence is a ρ-approximate common fixed pint sequence for a finite family of ρ-nonexpansive multivalued mappings. Under certain mild conditions, it ρ-converges to a common fixed point in modular function spaces.

2. Preliminaries

Now, we recall some basic notions and facts about modular spaces as formulated by Kozlowski [10]. For more details the reader may consult [5, 11] and the references therein.

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathscr{P}$ be a nontrivial δ-ring of subsets of Ω such that $E \cap A \in \mathscr{P}$ for any $E \in \mathscr{P}$ and $A \in \Sigma$. Assume that there exists an increasing sequence of sets $K_n \in \mathscr{P}$ such that $\Omega = \cup_{n=1} K_n$. By $\mathscr{E}$ we denote the linear space of all simple functions with supports from $\mathscr{P}$. By $\mathscr{M}_\infty$ we denote the space of all extended measurable functions, that is, all functions $f : \Omega \longrightarrow [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \mathscr{E}$, $|g_n| \leq |f|$ and $g_n(w) \longrightarrow f(w)$ for all $w \in \Omega$. By χ_A we denote the characteristic function of the set A.

Definition 2.1. Let $\rho : \mathscr{M}_\infty \longrightarrow [0, \infty]$ be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudo-modular if it satisfies the following:

a) $\rho(0) = 0$;
b) ρ is monotone; that is, $|f(w)| \leq |g(w)|$ for all $w \in \Omega$ implies $\rho(f) \leq \rho(g)$ where $f, g \in \mathscr{M}_\infty$;
c) ρ is orthogonally subadditive; that is, $\rho(f\chi_{A\cup B}) \leq \rho(f\chi_A) + \rho(f\chi_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, where $f \in \mathscr{M}_\infty$;
d) ρ has Fatou property; that is, $|f_n(w)| \uparrow |f(w)|$ for all $w \in \Omega$ implies that $\rho(f_n) \uparrow \rho(f)$ where $f \in \mathscr{M}_\infty$ and
e) ρ is order continuous in $\mathscr{E}$; that is, $g_n \in \mathscr{E}$ and $|g_n| \downarrow 0$ implies that $\rho(g_n) \downarrow 0$.

We say that a set $A \in \Sigma$ is ρ-null if $\rho(g\chi_A) = 0$ for every $g \in \mathscr{E}$. We say that a property p holds ρ-almost every where if the exceptional set $\{w \in \Omega : p(w) \textit{ does not hold}\}$ is ρ-null. As usual we identify any pair of measurable functions f *and* g differing only on ρ-null set by $f = g$ ρ-a.e. With this in mind we define

$$\mathscr{M} = \{f \in \mathscr{M}_\infty : |f(w)| < \infty \ \rho - a.e\},$$

where $f \in \mathscr{M}$ is actually an equivalence class of functions equal ρ-a.e rather than an individual function.

Definition 2.2. Let ρ be a regular convex function pseudo-modular.

a) We say that ρ is a regular convex function semi-modular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies that $f = 0$ ρ-a.e.

b) We say that ρ is a regular convex function modular if $\rho(f) = 0$ implies that $f = 0$ ρ-a.e.

The class of all nonzero regular convex function modulars defined on Ω is denoted by $\mathscr{R}$.

Remark 2.3. Let us denote $\rho(f,E) = \rho(f\chi_E)$ for $f \in \mathscr{M}, E \in \Sigma$. Also by convention for $\alpha > 0$ we will write $\rho(\alpha,E)$ instead of $\rho(\alpha\chi_E)$. We will use these notations when convenient. It is easy to prove that $\rho(f,E)$ is a convex function modular in the sense of Definition 2.1.

Remark 2.4. Note that if ρ is a regular convex function modular, then to verify that a set E is ρ-null it suffices to prove that there exists $\alpha > 0$ such that $\rho(\alpha,E) = 0$.

Definition 2.5. Let ρ be a convex function modular.

(1) A modular function space is the vector space $L_\rho(\Omega,\Sigma)$ or briefly L_ρ, defined by

$$L_\rho = \{f \in \mathscr{M} : \rho(\lambda f) \to 0 \; as \; \lambda \to 0\}.$$

(2) The following formula defines a norm in L_ρ frequently called the *Luxemburg norm*:

$$\|f\|_\rho = \inf\{\alpha > 0 : \rho(\frac{f}{\alpha}) \leq 1\}.$$

Definition 2.6. [11] Let $\rho \in \mathscr{R}$.

a) We say that $\{f_n\}$ is ρ-convergent to f and write $f_n \to f(\rho)$ if $\rho(f_n - f) \to 0$.

b) A sequence $\{f_n\}$ in L_ρ is called a ρ-Cauchy sequence if $\rho(f_n - f_m) \to 0$ as $n,m \to \infty$.

c) A set $B \subset L_\rho$ is called ρ-closed if for any sequence of $\{f_n\} \subset B$, the convergence $f_n \to f(\rho)$ implies that f belongs to B.

d) A set $B \subset L_\rho$ is called ρ-bounded if its ρ-diameter is finite; the ρ-diameter of B is defined as $\delta_\rho(B) = \sup\{\rho(f-g) : f \in B, g \in B\}$.

e) A set $B \subset L_\rho$ is called ρ-compact if for any $\{f_n\}$ in B, there exists a subsequence $\{f_{n_k}\}$ and an $f \in B$ such that $\rho(f_{n_k} - f) \to 0$.

f) A set $B \subset L_\rho$ is called ρ- a.e closed if if for any $\{f_n\}$ in B,which ρ-a.e converges to some f, we have $f \in B$.

g) A set $B \subset L_\rho$ is called ρ-a.e compact if for any $\{f_n\}$ in B, there exists a subsequence $\{f_{n_k}\}$ which ρ-a.e converges to some $f \in B$.

h) Let $f \in L_\rho$ *and* $B \subset L_\rho$. The ρ-distance between f *and* B is defined as $d_\rho(f,B) = \inf\{\rho(f-g) : g \in B\}$.

Theorem 2.7. [11] *Let $\rho \in \mathscr{R}$. L_ρ is complete with respect to ρ-convergence.*

The following definition plays very crucial role in modular function space and following this definition we get an important property that characterizes the convergence in function modular by the norm (Luxemburg norm)convergence (see the detail in [11]).

Definition 2.8. Let $\rho \in \mathscr{R}$. We say that ρ has the Δ_2 - property if $\rho(2f_n) \to 0$ whenever $\rho(f_n) \to 0$.

Proposition 2.9. [11] *The following statements are equivalent:*

(i) ρ *satisfies the* Δ_2*-condition.*

(ii) $\rho(f_n - f) \to 0$ *if and only if* $\rho(\lambda(f_n - f)) \to 0$*, for all* $\lambda > 0$ *if and only if* $\|f_n - f\|_\rho \to 0$.

Definition 2.10. [11] Let $\rho \in \mathscr{R}$. We say that ρ has the Δ_2 - type condition if there exists a constant $0 < k < \infty$ such that for every $f \in L_\rho$, we have $\rho(2f) \leq k\rho(f)$.

Remark 2.11. If ρ satisfies the Δ_2 - type condition, then it satisfies Δ_2 - property, and that the converse is not true (see, e.g.,[11]).

Let $\rho \in \mathscr{R}$ and C be a nonempty subset of the modular space L_ρ. We denote a collection of all nonempty ρ-closed and ρ- bounded subsets of C by $\mathscr{C}_\rho(C)$ and a collection of all nonempty ρ-compact subsets of C by $\mathscr{K}_\rho(C)$.

Definition 2.12. [6] A set $C \subset L_\rho$ is called ρ-proximinal if for each $f \in L_\rho$ there exists an element $g \in C$ such that

$$\rho(f-g) = d_\rho(f,C) = \inf\{\rho(f-h) : h \in C\}.$$

We denote the family of nonempty ρ-bounded ρ-proximinal subsets of C by $P_\rho(C)$.

Definition 2.13. [6] We define a Hausdorff distance on $\mathscr{C}_\rho(C)$ by,

$$H_\rho(A,B) = max\{\sup_{f\in A} d_\rho(f,B),\ \sup_{g\in B} d_\rho(g,A)\},$$

$A,B \in \mathscr{C}_\rho(C)$.

Definition 2.14. [6] A multivalued mapping $T : C \to \mathscr{C}_\rho(C)$ is called ρ-Lipschitzian if there exists a number $k \geq 0$ such that

$$H_\rho(T(f),T(g)) \leq k\rho(f-g)\ for\ all\ f,g \in C.$$

If $k \leq 1$ then, T is called ρ-nonexpansive and if $k < 1$, T is called ρ-contractive.

We find the following uniform convexity type property definitions of the function modular ρ in [5] and [8].

Definition 2.15. Let $\rho \in \mathscr{R}$. Let $t \in (0,1)$, $r > 0$, $\varepsilon > 0$. Define,

$$D_1(r,\varepsilon) = \{(f,g) : f,g \in L_\rho,\ \rho(f) \leq r,\ \rho(g) \leq r,\ \rho(f-g) \geq \varepsilon r\}.$$

Let

$$\delta_1^t(r,\varepsilon) = \inf\{1 - \frac{1}{r}\rho(tf + (1-t)g : (f,g) \in D_1(r,\varepsilon)\},\ if\ D_1(r,\varepsilon) \neq \emptyset$$

and $\delta_1^t(r,\varepsilon) = 1,\ if\ D_1(r,\varepsilon) = \emptyset$.
We will use the following notational convention: $\delta_1 = \delta_1^{\frac{1}{2}}$.

Definition 2.16. We say that ρ satisfies (UC1) if for every $r > 0,\ \varepsilon > 0,\ \delta_1(r,\varepsilon) > 0$. Note that for every $r > 0$, $D_1(r,\varepsilon) \neq \emptyset$, for $\varepsilon > 0$ small enough. We say that ρ satisfies (UUC1) if for every $s \geq 0,\ \varepsilon > 0$, there exists $\eta_1(s,\varepsilon) > 0$ depending only on s and ε such that

$$\delta_1(r,\varepsilon) > \eta_1(s,\varepsilon) > 0,\ for\ any\ r > s.$$

Definition 2.17. A sequence $\{t_n\} \subset (0,1)$ is called bounded away from 0 if there exists $0 < a < 1$ such that $t_n \geq a$, for every $n \in \mathbb{N}$. Similarly, $\{t_n\} \subset (0,1)$ is called bounded away from 1 if there exists $0 < b < 1$ such that $t_n \leq b$, for every $n \in \mathbb{N}$.

Lemma 2.18. [2] *Let ρ satisfies (UUC1) and let $\{t_n\} \subset (0,1)$ be bounded away from both 0 and 1. If there exists $R > 0$ such that*

$$\limsup_{n\to\infty} \rho(f_n) \leq R,\ \limsup_{n\to\infty} \rho(g_n) \leq R$$

and

$$\lim_{n\to\infty} \rho(t_n f_n + (1-t_n)g_n) = R,\ then$$

$$\lim_{n\to\infty} \rho(f_n - g_n) = 0.$$

Lemma 2.19. [6] *Let $T : C \to P_\rho(C)$ be a multi-valued mapping and*

$$P_\rho^T(f) = \{g \in Tf : \rho(f-g) = d_\rho(f,Tf)\}.$$

Then, the following are equivalent:

(i) $f \in F_\rho(T)$, *that is* $f \in Tf$;

(ii) $P_\rho^T(f) = \{f\}$, *that is,* $f = g$ *for each* $g \in P_\rho^T(f)$;

(iii) $f \in F_\rho(P_\rho^T(f))$, *that is,* $f \in P_\rho^T(f)$. *Further,* $F_\rho(T) = F(P_\rho^T)$ *where* $F(P_\rho^T)$ *denotes the set of fixed points of* P_ρ^T.

Definition 2.20. [4] A family of mappings $T_i : C \to C_\rho(C)$, $i = 1,2,...,m$, is said to satisfy Condition (II) if there exists a nondecreasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$, $\varphi(r) > 0$ for $r \in (0,\infty)$ such that

$$d_\rho(f,T_i(f)) \geq \varphi(d_\rho(f,\cap_{i=1}^m F_\rho(T_i)))$$

for some $i = 1,2,...,m$.

The following lemma also plays an important role in this article.

Lemma 2.21. [4] *Let $\rho \in \mathscr{R}$. Let $A,B \in P_\rho(L_\rho)$. For every $f \in A$, there exists $g \in B$ such that $\rho(f-g) \leq H_\rho(A,B)$.*

3. Main results

In what follows, we shall use the following iteration which is more general than that of the Mann iteration scheme used by Khan and Abbas [6]. Let $C \subset L_\rho$ be nonempty set and

$T_i : C \to P_\rho(C)$, $i = 1,2,...,m$, be a finite family of multi-valued mappings. Fix $f_1 \in C$ and define a sequence $\{f_n\} \subset C$ as follows:

$$f_{n+1} = \alpha_{n,0}f_n + \alpha_{n,1}g_{n,1} + \alpha_{n,2}g_{n,2} + ... + \alpha_{n,m}g_{n,m}, \tag{3.1}$$

where $g_{n,i} \in P_\rho^{T_i}(f_n)$ and $\{\alpha_{n,i}\} \subset (0,1)$ is bounded away from 0 and 1 such that $\sum_{i=0}^m \alpha_{n,i} = 1$.

Now, we are in a position to prove our first theorem.

Theorem 3.1. *Let $\rho \in \mathscr{R}$ satisfy (UUC1) and Δ_2-property. Let $C \subset L_\rho$ be ρ-closed, ρ-bounded and convex set. Suppose $T_i : C \to P_\rho(C)$, $i = 1,2,...,m$, be a finite family of multi-valued mappings such that $P_\rho^{T_i}$ is ρ-nonexpansive mapping for each $i = 1,2,...,m$. Assume that $F := \cap_{i=1}^m F_\rho(T_i) \neq \emptyset$. Let $\{f_n\}$ be as defined in (3.1). Then,*

(1) $\lim_{n\to\infty} \rho(f_n - p)$ *exists for all $p \in F$.*

(2) $\lim_{n\to\infty} d_\rho(f_n, T_i(f_n)) = 0$*, for all $i = 1,2,...,m$.*

Proof. Let $p \in F$ be arbitrary. Then, $P_\rho^{T_i}(p) = \{p\}$, $\forall i \in \{1,2,...,m\}$. From equation (3.1), we have

$$\begin{aligned} &\rho(f_{n+1} - p) \\ &= \rho(\alpha_{n,0}f_n + \alpha_{n,1}g_{n,1} + \alpha_{n,2}g_{n,2} + ... + \alpha_{n,m}g_{n,m} - p) \\ &= \rho(\alpha_{n,0}(f_n - p) + \alpha_{n,1}(g_{n,1} - p) + \alpha_{n,2}(g_{n,2} - p) + ... + \alpha_{n,m}(g_{n,m} - p)) \\ &\leq \alpha_{n,0}\rho(f_n - p) + \alpha_{n,1}\rho(g_{n,1} - p) + \alpha_{n,2}\rho(g_{n,2} - p) + ... + \alpha_{n,m}\rho(g_{n,m} - p). \end{aligned} \tag{3.2}$$

Since $g_{n,i} \in P_\rho^{T_i}(f_n)$ and $P_\rho^{T_i}$ is ρ-nonexpansive for all $i = 1,2,...,m$, we have

$$\begin{aligned} \rho(g_{n,i} - p) &\leq H_\rho(P_\rho^{T_i}(f_n), P_\rho^{T_i}(p)) \\ &\leq \rho(f_n - p) \end{aligned} \tag{3.3}$$

for all $i = 1,2,...,m$. Now, from (3.2), (3.3) and the assumption $\sum_{i=0}^m \alpha_{n,i} = 1$, we obtain that

$$\rho(f_{n+1} - p) \leq \rho(f_n - p), \quad \forall p \in F.$$

Therefore, $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in F$. Let

$$\lim_{n\to\infty} \rho(f_n - p) = \quad r \tag{3.4}$$

for some $r \geq 0$. From (3.3) and (3.4), we have

$$\limsup_{n\to\infty} \rho(g_{n,m} - p) \quad \leq r. \tag{3.5}$$

Consider

$$\begin{aligned}
&\rho\Big(\frac{\alpha_{n,0}}{1-\alpha_{n,m}}(f_n - p) + \frac{\sum_{i=1}^{m-1} \alpha_{n,i}(g_{n,i} - p)}{1-\alpha_{n,m}}\Big) \\
&\leq \frac{\alpha_{n,0}}{1-\alpha_{n,m}}\rho(f_n - p) + \frac{\sum_{i=1}^{m-1} \alpha_{n,i}\rho(g_{n,i} - p)}{1-\alpha_{n,m}} \\
&\leq \frac{\sum_{i=0}^{m-1} \alpha_{n,i}}{1-\alpha_{n,m}}\rho(f_n - p) \\
&= \rho(f_n - p).
\end{aligned}$$

Therefore, from (3.4) we obtain

$$\limsup_{n\to\infty} \rho\Big(\frac{\alpha_{n,0}}{1-\alpha_{n,m}}(f_n - p) + \frac{\sum_{i=1}^{m-1} \alpha_{n,i}(g_{n,i} - p)}{1-\alpha_{n,m}}\Big) \leq r. \tag{3.6}$$

Thus (3.4), (3.5), (3.6) and Lemma 2.18, give that

$$\lim_{n\to\infty} \rho\Big(\frac{\alpha_{n,0}}{1-\alpha_{n,m}} f_n + \frac{\sum_{i=1}^{m-1} \alpha_{n,i} g_{n,i}}{1-\alpha_{n,m}} - g_{n,m}\Big) \quad =0. \tag{3.7}$$

Now,

$$\begin{aligned}
\rho(f_{n+1} - g_{n,m}) &= \rho\Big(\alpha_{n,0} f_n + \sum_{i=1}^{m} \alpha_{n,i} g_{n,i} - g_{n,m}\Big) \\
&= \rho\Big(\alpha_{n,0} f_n + \sum_{i=1}^{m-1} \alpha_{n,i} g_{n,i} - (1-\alpha_{n,m}) g_{n,m}\Big) \\
&= \rho\Big((1-\alpha_{n,m})\Big[\frac{\alpha_{n,0}}{1-\alpha_{n,m}} f_n + \frac{\sum_{i=1}^{m-1} \alpha_{n,i} g_{n,i}}{1-\alpha_{n,m}} - g_{n,m}\Big]\Big).
\end{aligned}$$

From (3.7) by the Δ_2- property of ρ and Proposition 2.9, we get that

$$\lim_{n\to\infty} \rho(f_{n+1} - g_{n,m}) = 0.$$

In the same way, we can show that

$$\lim_{n\to\infty} \rho(f_{n+1} - f_n) = 0$$

and

$$\lim_{n\to\infty} \rho(f_{n+1} - g_{n,i}) = 0,$$

for all $i = 1,2,...,m-1$. Now by the convexity of ρ, we get

$$\begin{aligned}\rho(\frac{f_n - g_{n,i}}{2}) &= \rho(\frac{f_n - f_{n+1}}{2} + \frac{f_{n+1} - g_{n,i}}{2}) \\ &\leq \frac{\rho(f_n - f_{n+1})}{2} + \frac{\rho(f_{n+1} - g_{n,i})}{2}.\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty} \rho(\frac{f_n - g_{n,i}}{2}) = 0.$$

Since ρ satisfies Δ_2-property by Proposition 2.9, we obtain

$$\lim_{n\to\infty} \rho(f_n - g_{n,i}) = 0, \tag{3.8}$$

for all $i = 1,2,...,m$. Since $g_{n,i} \in P_\rho^{T_i}(f_n)$, we have

$$d_\rho(f_n, T_i(f_n)) = \rho(f_n - g_{n,i})$$

for all $i = 1,2,...,m$. Therefore,

$$\lim_{n\to\infty} d_\rho(f_n, T_i(f_n)) = 0$$

follows immediately from (3.8) for all $i = 1,2,...,m$.

Theorem 3.2. *Let $\rho \in \mathscr{R}$ satisfy (UUC1) and Δ_2-property. Let $C \subset L_\rho$ be ρ-closed, ρ-bounded and convex set. Suppose $T_i : C \to P_\rho(C)$, $i = 1,2,...,m$, be a finite family of multi-valued mappings such that $P_\rho^{T_i}$ is ρ-nonexpansive mapping for each $i = 1,2,...,m$. Let $F := \cap_{i=1}^m F_\rho(T_i) \neq \emptyset$ and $\{f_n\}$ be as defined in equation (3.1). Then, $\{f_n\}$ ρ-converges to a point in F if and only if* $\liminf_{n\to\infty} d_\rho(f_n, F) = 0$.

Proof. The necessity is straight forward. Now, we prove the other way round. Suppose that $\liminf_{n\to\infty} d_\rho(f_n, F) = 0$. By Theorem 3.1, we have

$$\rho(f_{n+1} - p) \leq \rho(f_n - p), \text{ for all p } \in F.$$

This rises to

$$d_\rho(f_{n+1}, F) \leq d_\rho(f_n, F).$$

Hence, $\lim_{n\to\infty} d_\rho(f_n, F)$ exists. But by hypothesis, $\liminf_{n\to\infty} d_\rho(f_n, F) = 0$. Therefore, it must be the case that

$$\lim_{n\to\infty} d_\rho(f_n, F) = 0.$$

Next we show that $\{f_n\}$ is a ρ-Cauchy sequence in C. Let $\varepsilon > 0$ be arbitrary. Then there exists an integer $n_0 \in \mathbb{N}$ such that

$$d_\rho(f_n, F) < \frac{\varepsilon}{2}, \ \forall n \geq n_0.$$

In particular, $inf\{\rho(f_{n_0} - p) : p \in F\} < \frac{\varepsilon}{2}$. Thus, there must exist a $p_0 \in F$ such that

$$\rho(f_{n_0} - p_0) < \varepsilon.$$

Now for $m, n \geq n_0$, we have

$$\begin{aligned}
\rho(\frac{f_m - f_n}{2}) &\leq \frac{1}{2}\rho(f_m - p_0) + \frac{1}{2}\rho(f_n - p_0) \\
&\leq \frac{\rho(f_{n_0} - p_0)}{2} + \frac{\rho(f_{n_0} - p_0}{2}) \\
&< \varepsilon.
\end{aligned}$$

Since ρ satisfies Δ_2 condition, by Proposition 2.9 we get $\{f_n\}$ is a ρ-Cauchy sequence in C. Since L_ρ is complete with respect to ρ-convergence and C is ρ-closed , there exists an $f \in C$ such that $\rho(f_n - f) \to 0$.

Next, we show that f is a common fixed point of $\{T_i\}_{i=1}^m$. Let $g_i \in P_\rho^{T_i}(f)$ be arbitrary. By Lemma 2.21, there exits $g_{n,i} \in P_\rho^{T_i}(f_n)$ such that $\rho(g_{n,i} - g_i) \leq H_\rho(P_\rho^{T_i}(f_n), P_\rho^{T_i}(f))$ for all $i = 1, 2, ..., m$. Then by convexity of ρ and Theorem 3.1, we have

$$\begin{aligned}
\rho(\frac{f - g_i}{3}) &= \rho(\frac{f - f_n}{3} + \frac{f_n - g_{n,i}}{3} + \frac{g_{n,i} - g_i}{3}) \\
&\leq \frac{1}{3}\rho(f - f_n) + \frac{1}{3}\rho(f_n - g_{n,i}) + \frac{1}{3}\rho(g_{n,i} - g_i) \\
&\leq \rho(f - f_n) + d_\rho(f_n, P_\rho^{T_i}(f_n)) + H_\rho(P_\rho^{T_i}(f_n), P_\rho^{T_i}(f)) \\
&\leq \rho(f - f_n) + d_\rho(f_n, T_i(f_n)) + \rho(f - f_n) \to 0 \text{ as n } \to \infty.
\end{aligned}$$

Thus, $f = g_i$. Since $g_i \in P_\rho^{T_i}(f)$ by Lemma 2.19, we have $f \in T_i(f)$ for all i= 1,2,...,m . Therefore, $f \in F$, i.e., $\{f_n\}$ ρ-converges to a common fixed point of the family $\{T_i\}_{i=1}^m$.

Corollary 3.3. *Let $\rho \in \mathscr{R}$ satisfy (UUC1) and Δ_2-property. Let $C \subset L_\rho$ be ρ-closed, ρ-bounded and convex set. Suppose $T_i : C \to P_\rho(C)$, $i = 1, 2, ..., m$, be a finite family of multi-valued mappings satisfying Condition (II) such that $P_\rho^{T_i}$ is ρ-nonexpansive mapping for each $i = 1, 2, ..., m$. Assumethat $F := \cap_{i=1}^m F_\rho(T_i) \neq \emptyset$. Let $\{f_n\}$ be as defined in equation (3.1). Then $\{f_n\}$ ρ-converges to a point in $F := \cap_{i=1}^m F_\rho(T_i)$.*

Proof. By Theorem 3.1, $\lim_{n\to\infty}\rho(f_n-p)$ exists for all $p\in F$. If $\lim_{n\to\infty}\rho(f_n-p)=0$, there is nothing to prove. Suppose $\lim_{n\to\infty}\rho(f_n-p)=R>0$. Again from Theorem 3.1, we have

$$\rho(f_{n+1}-p)\leq\rho(f_n-p).$$

Thus,

$$d_\rho(f_{n+1},F)\leq d_\rho(f_n,F)\quad\forall p\in F.$$

Hence, $\lim_{n\to\infty}d_\rho(f_n,F)$ exists. By Condition (II) and Theorem 3.1, we have

$$0=\lim_{n\to\infty}d_\rho(f_n,T_i(f_n))\geq\lim_{n\to\infty}\varphi(d_\rho(f_n,F))$$

for some $i=1,2,...,m$. Thus, $\lim_{n\to\infty}\varphi(d_\rho(f_n,F))=0$. Since φ is nondecreasing and $\varphi(0)=0$, $\lim_{n\to\infty}d_\rho(f_n,F)=0$. By Theorem 3.2, we find the desired result immediately.

Remark 3.4. If, in the iteration scheme (3.1), we consider $m=1$, then the scheme coincides with the Mann iteration scheme in Theorem 1.1 used by Khan and Abbas [6].

Acknowledgements

This work was supported by Simmons Foundation based at Botswana International University of Science and Technology. The second author was also supported in part by the International Science Program (ISP)- Sweden.

References

[1] A.A.N. Abdoul, M.A. Khamsi, A.R. Khan, Convergence of Ishikawa iterates of two mappings in modular function spaces, Fixed Point Theory Appl. 2014 (2014), Article ID 74.

[2] B.A.B. Dehaish, W.M. Kozlowski, Fixed point iteration processes for asymptotic pointwise nonexpansive mapping in modular function spaces, Fixed Point Theory Appl. 2012 (2012), Article ID 118.

[3] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44 (1974), 147-150.

[4] W.W. Kassu, M.G. Sangago, H. Zegeye, Approximating common fixed points of two multi-valued mappings by Ishikawa iterates in modular function spaces, Int. J. Nonlinear Anal. Appl. submitted.

[5] M.A. Khamsi, W.M. Kozlowski, Fixed point theory in modular function spaces, Springer International Publishing Switzerland, 2015.

[6] S.H. Khan, M. Abbas, Approximating fixed points of multivalued ρ-nonexpansive mappings in modular function spaces, Fixed Point Theory Appl. 2014 (2014), Article ID 34.

[7] M.A. Khamsi, W.M. Kozlowski, On asymptotic pointwise nonexpansive mappings in modular function s-paces, J.Math. Anal. Appl. 380 (2011), 697-708.

[8] M.A. Khamsi, A convexity property in modular function spaces, Math. Japon. 44 (1996), 269-279.

[9] M.A. Khamsi, W.M. Kozlowsk, S. Reich, Fixed Point Theory in Modular Function Spaces, Nonlinear Anal. 14 (1990), 935-953.

[10] W. M. Kozlowski, Advancements in fixed point theory in modular function spaces, Arab. J. Math. 1 (2012), 477-494.

[11] W.M. Kozlowski, Modular function spaces, Monographs and Textbooks in Pure and Applied Mathematics. Volume 122, Marcel Dekker, New York, USA, 1988.

[12] P. Kumam, Fixed point theorem for non-expansive mappings in modular spaces, Arch. Math. 40 (2004), 345-353.

[13] M.A. Kutib, A. Latif, Fixed points of multivalued maps in modular function spaces, Fixed Point Theory Appl. 2009 (2009), Article ID 786357.

[14] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 44 (1953), 506-510.

[15] J. Musielak and W. Orlicz, On modular spaces, Stud. Math. 18 (1959), 591-597.