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A GENERAL VISCOSITY ITERATIVE ALGORITHM FOR MONOTONE VARIATIONAL INEQUALITY AND GENERALIZED EQUILIBRIUM PROBLEMS

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Abstract. In this article, we consider a general viscosity iterative algorithm for monotone variational inequality and generalized equilibrium problems. Strong convergence of the general viscosity iterative algorithm is obtained.

Keywords: Equilibrium problem; Fixed point; Monotone operator; Viscosity approximation method; Zero point.

1. Introduction

Equilibrium problems and monotone variational inequality theory is dynamic and is experiencing an explosive growth in both theory and applications as a consequence, research techniques and problems are drawn from various fields. Monotone variational inequality and equilibrium problems have been generalized and extended in different directions using the novel and innovative techniques. Mann's iterative algorithm is efficient to study the problems. However, in an infinite-dimensional Hilbert space, Mann's iteration algorithms has only weak convergence, in general [1]. In many disciplines, including economics [2], image recovery [3], quantum physics [4], and control theory [5], problems arises in infinite dimension spaces. In such problems, strong convergence (norm convergence) is often much more desirable than weak convergence, for it translates the physically tangible property that the energy $\|x_n - x\|$ of the error between the iterate x_n and the solution x eventually becomes arbitrarily small. In this paper, we investigate a general viscosity iterative algorithms for treating monotone variational inequality

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and generalized equilibrium problems. Strong convergence theorems are established. The organization of this article is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, strong convergence analysis of the general viscosity iterative algorithm is provided. Some corollaries of the main results are also provided.

2. Preliminaries

Throughout this paper, Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. Let C be a nonempty closed and convex subset of H and let P_C be the metric projection of H onto C. Let $T : C \to C$ be a mapping. We use $F(T)$ to denote the fixed point set of T. Recall that T is said to be α-contractive iff there exists a constant $\alpha \in (0,1)$ such that

$$\|Tx - Ty\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.$$

Recall that T is said to be nonexpansive iff

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

If C is also bounded, then we the fixed point set of T is nonempty. Let $A : C \to H$ be a map. The classical variational inequality problem which denoted by $VI(C,A)$ is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C. \tag{2.1}$$

It is known that P_C is firmly nonexpansive, that is,

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.2}$$

for Moreover, $P_C x$ is characterized by the properties: $P_C x \in C$ and $\langle x - P_C x, P_C x - y\rangle \geq 0$ for all $y \in C$. One can see that the variational inequality (2.1) is equivalent to a fixed point problem. The element $u \in C$ is a solution of the variational inequality (2.1) if and only if $u \in C$ satisfies the relation $u = P_C(u - \lambda Au)$, where $\lambda > 0$ is a constant. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems; see, for example, [6-13] and the references therein.

Recall that A is said to be α-inverse-strongly monotone iff there exists a positive number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is obivous if A is α-inverse-strongly monotone, then A is continuous and monotone.

Recall that an operator A is strong positive on H if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let B be a monotone map of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \ \forall u \in C\}$ and define

$$Tv = \begin{cases} Bv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, B)$; for more details, see [14] and the reference therein.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{2.3}$$

The set of solution of (2.3) is denoted by $EP(F)$. Give a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $z \in EP(F)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. Numerous problems in physics, optimization and economics reduce to find a solution of (2.3). Some methods have been proposed to solve the equilibrium problem; see, for instance, [15-22] and the references therein.

In 2007, Takahashi *et al.* [22] introduced a iterative scheme: $x_0 \in C$ and

$$\begin{cases} F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Ty_n, \quad n \geq 0, \end{cases} \tag{2.4}$$

where f is a contraction on H, F is a bifunction, and T is a nonexpansive maping for approximating a common element of the set of fixed points of a non-self nonexpansive mapping and the set of solutions of the equilibrium problem and obtained a strong convergence theorem in a real Hilbert space.

In [21], Su *et al.* improved the results of [22] and studied the following iterative algorithm: $x_0 \in H$

$$\begin{cases} F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)TP_C(I - \lambda_n A)y_n, \quad n \geq 0, \end{cases} \tag{2.5}$$

where f is a contraction on H, F is a bifunction and A is inverse-strongly monotone operator of C into H, T is a nonexpansive mapping. They proved the sequence $\{x_n\}$ defined by above iterative algorithm converges strongly to a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of the equilibrium problems and the set of solutions of variational inequality problems.

In 2008, S. Takahahsi and W. Takahashi [15] introduced and investigated the following generalized equilibrium problem. Find $x \in C$ such that

$$F(x,y) + \langle Bx, y - x\rangle \geq 0, \quad \forall y \in C, \tag{2.6}$$

where $B : C \to H$ is a monotone operator. The set of solution of (2.3) is denoted by $GEP(F,B)$. If $F(x,y) = 0$, $\forall x,y \in C$, the generalized equilibrium problem (2.6) is reduced to monotone variational inequality (2.1). If $Ax = 0$, $\forall x \in C$, the generalized equilibrium problem (2.6) is reduced to equilibrium problem (2.3).

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in F(S)} \frac{1}{2}\langle Mx, x\rangle - \langle x, b\rangle, \tag{2.6}$$

where M is a linear bounded operator, $F(S)$ is the fixed point set of a nonexpansive mapping S and b is a given point in H. In [23], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n M)Sx_n + \alpha_n b, \ \ n \geq 0,$$

converges strongly to the unique solution of the minimization problem (2.6) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Recently, Marino and Xu [24] introduced a new iterative scheme by the viscosity approximation method [25]:

$$x_0 \in H, \quad x_{n+1} = (I - \alpha_n M)Sx_n + \alpha_n \gamma f(x_n), \ \ n \geq 0.$$

They proved the sequence $\{x_n\}$ generated by above iterative scheme converges strongly to the unique solution of the variational inequality $\langle (M - \gamma f)x^*, x - x^* \rangle \geq 0$, $x \in C$, which is the optimality condition for the minimization problem $\min_{x \in F(S)} \frac{1}{2}\langle Mx, x \rangle - h(x)$, where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

Recall that A space X is said to satisfy Opial's condition [26] if for each sequence $\{x_n\}_{n=1}^{\infty}$ in X which converges weakly to point $x \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \ \forall y \in X, \ y \neq x.$$

Lemma 2.1. [27] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - t_n)a_n + b_n + c_n$, $\forall n \geq 0$, where $\{c_n\}$ is a sequence of nonnegative real numbers, $\{t_n\} \subset (0,1)$ and $\{b_n\}$ is a sequence of real numbers. Assume that*

(a) $\sum_{n=0}^{\infty} t_n = \infty$ *and* $\limsup_{n\to\infty} \frac{b_n}{t_n} \leq 0$;

(b) $\sum_{n=0}^{\infty} c_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. [29] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let β_n be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.3. [24] *Assume M is a strong positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \le \|M\|^{-1}$. Then $\|I - \rho M\| \le 1 - \rho\bar{\gamma}$.*

Lemma 2.4. *Let C be a nonempty closed convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)$-$(A4)$. Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C\}$$

for all $r > 0$ and $x \in H$. Then, the following hold:

(a) *T_r is single-valued;*

(b) *T_r is firmly nonexpansive;*

(c) *$Fix(T_r) = EP(F)$;*

(d) *$EP(F)$ is closed and convex.*

Lemma 2.5. *Let C, H, F and T_r be as in Lemma 2.2. Then the following holds:*

$$\|T_r x - T_s x\|^2 \le \frac{s-r}{s}\langle T_s x - T_r x, T_s x - x\rangle, \forall r, s > 0, x \in H.$$

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $B : C \to H$ be a β-inverse-strongly monotone mapping. Let f be a contraction of H into itself with a coefficient κ ($0 < \kappa < 1$) and let M*

be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ with $0 < \gamma < \bar{\gamma}/\kappa$. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1), (A2), (A3) and (A4). Assume that $\Omega = GEP(F,A) \cap VI(C,B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$\begin{cases} r_n F(z_n,\eta) + r_n\langle Ax_n, \eta - z_n\rangle + \langle \eta - z_n, z_n - x_n\rangle \geq 0, \forall \eta \in C, \\ x_{n+1} = \beta_n(1-\alpha_n)\gamma f(z_n) + (1-\alpha_n)(I-\beta_n M)P_C(z_n - s_n B z_n) + \alpha_n x_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequence in $(0,1)$ and $\{r_n\},\{s_n\} \subset (0,\infty)$. Assume that the control sequences satisfy the following conditions: $\lim_{n\to\infty}\beta_n = 0, \sum_{n=1}^{\infty}\beta_n = \infty$, $\lim_{n\to\infty}|r_{n+1} - r_n| = 0, \lim_{n\to\infty}|s_{n+1} - s_n| = 0$, $0 < a < \alpha_n < b < 1$, $0 < c \leq s_n \leq d < 2\alpha$ and $0 < e \leq r_n \leq f < 2\beta$, where a,b,c,d,e and f are constants. Then, $\{x_n\}$ converges strongly to $q \in \Omega$, where $q = P_\Omega(\gamma f + (I-M))(q)$, which solves the following variational inequality $\langle \gamma f(q) - Mq, p - q\rangle \leq 0, \ \forall p \in \Omega$.

Proof. First, we show $I - s_n B$ and $I - r_n A$ are nonexpansive. For any $x, y \in C$, we see that

$$\begin{aligned} &\|(I-r_nA)x - (I-r_nA)y\|^2 \\ &= \|x-y\|^2 - 2r_n\langle x-y, Ax-Ay\rangle + r_n^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 - r_n(2\alpha - r_n)\|Ax-Ay\|^2. \end{aligned}$$

By using the condition imposed on $\{r_n\}$, we see that $\|(I-r_nA)x - (I-r_nA)y\| \leq \|x-y\|$. This proves that $I - r_nA$ is nonexpansive. In the same way, we find that $I - s_nB$ is also nonexpansive. Now, we observe that $\{x_n\}$ is bounded. Fix $p \in \Omega$ and set $\rho_n = P_C(I - s_nB)z_n$. Since $\beta_n \to 0$, we may assume, with no loss of generality, that $\beta_n < \|M\|^{-1}$ for all n. From Lemma 2.3, we know that if $0 < \rho \leq \|M\|^{-1}$, then $\|I - \rho M\| \leq 1 - \rho\bar{\gamma}$. Note that

$$\begin{aligned} \|z_n - p\| \leq &\|T_n(x_n - r_nAx_n) - T_n(p - r_nAp)\| \\ &\leq \|(x_n - r_nAx_n) - (p - r_nAp)\| \\ &\leq \|x_n - p\|^2 - r_n(2\alpha - r_n)\|Ax_n - Ap\|^2. \end{aligned} \tag{3.1}$$

Putting $y_n = \beta_n \gamma f(z_n) + (I - \beta_n M)\rho_n$, we have

$$\begin{aligned}\|y_n - p\| \leq& \beta_n \|\gamma f(z_n) - Mp\| + \|I - \beta_n M\| \|\rho_n - p\| \\ \leq& \beta_n[\gamma \|f(z_n) - f(p)\| + \|\gamma f(p) - Mp\|] + (1 - \beta_n \bar{\gamma}) \|\rho_n - p\| \\ \leq& \beta_n[\gamma \|f(z_n) - f(p)\| + \|\gamma f(p) - Mp\|] + (1 - \beta_n \bar{\gamma}) \|z_n - p\| \\ \leq& [1 - (\bar{\gamma} - \gamma\kappa)\beta_n] \|z_n - p\| + \beta_n \|\gamma f(p) - Mp\|.\end{aligned}$$

This yields from (3.1) that

$$\begin{aligned}\|x_{n+1} - p\| &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|y_n - p\| \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n)[1 - (\bar{\gamma} - \gamma\kappa)\beta_n] \|x_n - p\| + (1 - \alpha_n)\beta_n \|\gamma f(p) - Mp\| \\ &= \Big(1 - \beta_n(1 - \alpha_n)(\bar{\gamma} - \gamma\kappa)\Big) \|x_n - p\| + (1 - \alpha_n)\beta_n \|\gamma f(p) - Mp\|.\end{aligned}$$

This in turn implies that

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \ \frac{\|\gamma f(p) - Mp\|}{\bar{\gamma} - \gamma\kappa}\}, \quad n \geq 1.$$

Therefore, we obtain that $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. Note that

$$\begin{aligned}\|\rho_n - \rho_{n+1}\| &\leq \|(z_n - s_n B z_n) - (z_{n+1} - s_{n+1} B z_{n+1})\| \\ &\leq \|z_n - z_{n+1}\| + |s_n - s_{n+1}| \|B z_{n+1}\|\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\|z_n - z_{n+1}\| \leq& \|T_{r_n}(x_n - r_n A x_n) - T_{r_{n+1}}(x_{n+1} - r_{n+1} A x_{n+1})\| \\ \leq& \|(x_n - r_n A x_n) - (x_{n+1} - r_{n+1} A x_{n+1})\| \\ &+ \|T_{r_n}(x_{n+1} - r_{n+1} A x_{n+1}) - T_{r_{n+1}}(x_{n+1} - r_{n+1} A x_{n+1})\| \\ \leq& \|(x_n - r_n A x_n) - (x_{n+1} - r_{n+1} A x_{n+1})\| \\ &+ \frac{|r_{n+1} - r_n|}{r_{n+1}} \langle T_{r_{n+1}}(x_{n+1} - r_{n+1} A x_{n+1}) - T_{r_n}(x_{n+1} - r_{n+1} A x_{n+1}), \\ &T_{r_{n+1}}(x_{n+1} - r_{n+1} A x_{n+1}) - (x_{n+1} - r_{n+1} A x_{n+1})\rangle \\ \leq& \|x_n - x_{n+1}\| + |r_n - r_{n+1}| \|A x_{n+1}\| \\ &+ \frac{|r_{n+1} - r_n|}{r_{n+1}} \|T_{r_{n+1}} u_n - T_{r_n} u_n\| \|T_{r_{n+1}} u_n - u_n\|,\end{aligned} \tag{3.3}$$

where $u_n = (x_{n+1} - r_{n+1}Ax_{n+1})$. From (3.2) and (3.3), we have

$$\begin{aligned}
\|y_n - y_{n+1}\| &\le \beta_{n+1}\gamma\|f(z_{n+1}) - f(z_n)\| + (1-\beta_{n+1}\bar{\gamma})\|\rho_{n+1} - \rho_n\| \\
&\quad + (\gamma\|f(z_n)\| + \|M\rho_n\|)|\beta_{n+1} - \beta_n| \\
&\le \beta_{n+1}\gamma\kappa\|z_{n+1} - z_n\| + (1-\beta_{n+1}\bar{\gamma})\|\rho_{n+1} - \rho_n\| \\
&\quad + (\gamma\|f(z_n)\| + \|M\rho_n\|)|\beta_{n+1} - \beta_n| \\
&\le \Big(1 - \beta_n(\bar{\gamma} - \gamma\kappa)\Big)\|z_{n+1} - z_n\| \\
&\quad + |s_n - s_{n+1}|\|Bz_{n+1}\| + (\gamma\|f(z_n)\| + \|M\rho_n\|)|\beta_{n+1} - \beta_n| \\
&\le \Big(1 - \beta_n(\bar{\gamma} - \gamma\kappa)\Big)\|x_n - x_{n+1}\| + |r_n - r_{n+1}|R_1 \\
&\quad + |s_n - s_{n+1}|\|Bz_{n+1}\| + (\gamma\|f(z_n)\| + \|M\rho_n\|)|\beta_{n+1} - \beta_n|,
\end{aligned}$$

where R_1 is an appropriate constant such that $R_1 = \sup\{\|Ax_{n+1}\| + \frac{\|T_{r_{n+1}}u_n - T_{r_n}u_n\|\|T_{r_{n+1}}u_n - u_n\|}{e}\}$. It follows from the conditions that

$$\limsup_{n\to\infty}(\|y_n - y_{n+1}\| - \|x_n - x_{n+1}\|) \le 0.$$

Hence, $\lim_{n\to\infty}\|x_n - y_n\| = 0$. It follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.4}$$

Since A is inverse-strongly monotone, we find that

$$\begin{aligned}
\|z_n - p\|^2 &\le \|(x_n - p) - r_n(Ax_n - Ap)\|^2 \\
&\le \|x_n - p\|^2 - r_n(2\alpha - r_n)\|Ax_n - Ap\|^2.
\end{aligned} \tag{3.5}$$

Since B is inverse-strongly monotone, we find from (3.5) that

$$\begin{aligned}
\|\rho_n - p\|^2 &\le \|(z_n - p) - s_n(Bz_n - Bp)\|^2 \\
&\le \|z_n - p\|^2 - s_n(2\beta - r_n)\|Bz_n - Bp\|^2 \\
&\le \|x_n - p\|^2 - r_n(2\alpha - r_n)\|Ax_n - Ap\|^2 - s_n(2\beta - r_n)\|Bz_n - Bp\|^2
\end{aligned}$$

which implies

$$
\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\beta_n\|\gamma f(z_n)-Ap\|^2+(1-\alpha_n)(1-\beta_n\bar{\gamma})\|\rho_n-p\|^2 \\
&\quad +2\beta_n\|\gamma f(z_n)-Ap\|\|\rho_n-p\| \\
&\le \alpha_n\|x_n-p\|^2+\beta_n\|\gamma f(z_n)-Ap\|^2+(1-\alpha_n)(1-\beta_n\bar{\gamma})\|x_n-p\|^2 \\
&\quad -(1-\alpha_n)(1-\beta_n\bar{\gamma})r_n(2\alpha-r_n)\|Ax_n-Ap\|^2-(1-\alpha_n)(1-\beta_n\bar{\gamma})s_n(2\beta-r_n)\|Bz_n-Bp\|^2 \\
&\quad +2\beta_n\|\gamma f(z_n)-Ap\|\|\rho_n-p\| \\
&\le \|x_n-p\|^2+\beta_n\|\gamma f(z_n)-Ap\|^2-(1-\alpha_n)(1-\beta_n\bar{\gamma})r_n(2\alpha-r_n)\|Ax_n-Ap\|^2 \\
&\quad -(1-\alpha_n)(1-\beta_n\bar{\gamma})s_n(2\beta-r_n)\|Bz_n-Bp\|^2 \\
&\quad +2\beta_n\|\gamma f(z_n)-Ap\|\|\rho_n-p\|.
\end{aligned}
$$

This finds from the restrictions imposed on the control sequences that

$$\lim_{n\to\infty}\|Ax_n-Ap\|=\lim_{n\to\infty}\|Bz_n-Bp\|. \tag{3.6}$$

Since P_C is firmly nonexpansive, we have

$$
\begin{aligned}
\|\rho_n-p\|^2 \le&\langle (I-s_nB)z_n-(I-s_nB)p,\rho_n-p\rangle \\
=&\frac{1}{2}\{\|(I-s_nB)z_n-(I-s_nB)p\|^2+\|\rho_n-p\|^2 \\
&-\|(I-s_nB)z_n-(I-s_nB)p-(\rho_n-p)\|^2\} \\
\le&\frac{1}{2}\{\|z_n-p\|^2+\|\rho_n-p\|^2-\|(z_n-\rho_n)-s_n(Bz_n-Bp)\|^2\} \\
=&\frac{1}{2}\{\|x_n-p\|^2+\|\rho_n-p\|^2-\|z_n-\rho_n\|^2-s_n^2\|Bz_n-Bp\|^2 \\
&+2s_n\langle z_n-\rho_n,Bz_n-Bp\rangle\},
\end{aligned}
$$

which yields that

$$\|\rho_n-p\|^2\le\|x_n-p\|^2-\|z_n-\rho_n\|^2+2s_n\|z_n-\rho_n\|\|Bz_n-Bp\|.$$

Hence, we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\beta_n\|\gamma f(z_n)-Ap\|^2+(1-\alpha_n)(1-\beta_n\bar{\gamma})\|\rho_n-p\|^2 \\
&\quad +2\beta_n\|\gamma f(z_n)-Ap\|\|\rho_n-p\| \\
&\le \|x_n-p\|^2+\beta_n\|\gamma f(z_n)-Ap\|^2-(1-\alpha_n)(1-\beta_n\bar{\gamma})\|z_n-\rho_n\|^2+2s_n\|z_n-\rho_n\|\|Bz_n-Bp\| \\
&\quad +2\beta_n\|\gamma f(z_n)-Ap\|\|\rho_n-p\|.
\end{aligned}$$

This yields from (3.4) and (3.6) that

$$\lim_{n\to\infty}\|z_n-\rho_n\|=0. \tag{3.7}$$

Since T_{r_n} is firmly nonexpansive, we have

$$\begin{aligned}
\|z_n-p\|^2 \le&\langle (I-r_nA)x_n-(I-r_nA)p, z_n-p\rangle \\
=&\frac{1}{2}\{\|(I-r_nA)x_n-(I-r_nA)p\|^2+\|z_n-p\|^2 \\
&-\|(I-r_nA)x_n-(I-r_nA)p-(z_n-p)\|^2\} \\
\le&\frac{1}{2}\{\|x_n-p\|^2+\|z_n-p\|^2-\|(x_n-z_n)-r_n(Ax_n-Ap)\|^2\} \\
=&\frac{1}{2}\{\|x_n-p\|^2+\|z_n-p\|^2-\|x_n-z_n\|^2-r_n^2\|Ax_n-Ap\|^2 \\
&+2s_n\langle x_n-z_n, Ax_n-Ap\rangle\},
\end{aligned}$$

which yields that

$$\|z_n-p\|^2 \le \|x_n-p\|^2-\|x_n-z_n\|^2+2r_n\|x_n-z_n\|\|Ax_n-Ap\|.$$

Hence, we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\beta_n\|\gamma f(z_n)-Ap\|^2+(1-\alpha_n)(1-\beta_n\bar{\gamma})\|z_n-p\|^2 \\
&\quad +2\beta_n\|\gamma f(z_n)-Ap\|\|\rho_n-p\| \\
&\le \|x_n-p\|^2+\beta_n\|\gamma f(z_n)-Ap\|^2-(1-\alpha_n)(1-\beta_n\bar{\gamma})\|x_n-z_n\|^2+2r_n\|x_n-z_n\|\|Ax_n-Ap\| \\
&\quad +2\beta_n\|\gamma f(z_n)-Ap\|\|\rho_n-p\|.
\end{aligned}$$

This yields from (3.4) and (3.6) that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.8}$$

Now, we are in a position to prove that

$$\limsup_{n\to\infty}\langle x_n - q, (\gamma f - A)q\rangle \le 0,$$

where $q = P_\Omega[I-(M-\gamma f)]q$. To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n - q, (\gamma f - M)q\rangle = \lim_{i\to\infty}\langle x_{n_i} - q, (\gamma f - M)q\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to ξ. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup \xi$.

Next, we show $\xi \in \Omega$. First, we prove $\xi \in GEP(F,A)$. Since $z_n = T_{r_n}(I - r_nA)x_n$, we have

$$r_nF(z_n,z) + r_n\langle Ax_n, z - z_n\rangle + \langle z - z_n, z_n - x_n\rangle \ge 0, \quad \forall z \in C.$$

By use of condition (A2), we see that

$$\langle Ax_n, z - z_n\rangle + \langle z - z_n, \frac{z_n - x_n}{r_n}\rangle \ge F(z,z_n), \quad \forall z \in C.$$

For t with $0 < t \le 1$, and $z \in C$, let $z_t = tz + (1-t)\xi$. Since $y \in C$, and $\xi \in C$, we have $z_t \in C$. Hence, we have

$$\begin{aligned}
\langle z_t - z_n, Az_t\rangle &\ge \langle z_t - z_n, Az_t\rangle - \langle Ax_n, z_t - z_n\rangle - \langle z_t - z_n, \frac{z_n - x_n}{t_n}\rangle + F(z_t,z_n)\\
&\ge \langle z_t - z_n, Az_t - Az_n\rangle + \langle z_t - z_n, Az_n - Ax_n\rangle - \langle z_t - z_n, \frac{z_n - x_n}{t_n}\rangle + F(z_t,z_n)\\
&\ge \langle z_t - z_n, Az_n - Ax_n\rangle - \langle z_t - z_n, \frac{z_n - x_n}{t_n}\rangle + F(z_t,z_n).
\end{aligned}$$

Since $\{z_{n_i}\}$ converges weakly to ξ, we find that

$$\langle z_t - \xi, Az_t\rangle \ge F(z_t,\xi),$$

which implies that

$$\begin{aligned}
0 =& F(z_t,z_t) \le tF(z_t,y) + (1-t)F(z_t,\xi)\\
&\le tF(z_t,z) + (1-t)\langle z_t - \xi, Az_t\rangle\\
&= tF(z_t,z) + (1-t)t\langle z - \xi, Az_t\rangle,
\end{aligned}$$

that is, $0 \le F(z_t,z)+(1-t)\langle z-\xi, Az_t\rangle$. Letting $t\to 0$, we have $0\le F(\xi,z)+\langle z-\xi, A\xi\rangle$. This implies that $\xi \in GEP(F,A)$.

Next, we show $\xi \in VI(C,B)$. Put $S\lambda = N_C + B\lambda$, $\lambda \in C$ and $S\lambda = \emptyset$, $\lambda \notin C$. Since B is a monotone operator, we see that S is a maximal monotone operator. Let $(\lambda,\lambda') \in Graph(S)$. Since $\lambda' - B\lambda \in N_C\lambda$ and $\lambda_n \in C$, we have $\langle \lambda-\rho_n, \lambda' - B\lambda\rangle \ge 0$. On the other hand, we have from $\rho_n = Proj_C(z_n - s_nBz_n)$ that $\langle \xi-\rho_n, \rho_n-(I-s_nB)z_n\rangle \ge 0$ That is,

$$\langle \lambda-\rho_n, \frac{\rho_n - z_n}{s_n} + Bz_n\rangle \ge 0.$$

It follows from the above that

$$\begin{aligned}
\langle \lambda-\rho_{n_i}, \lambda'\rangle &\ge \langle \lambda-\rho_{n_i}, B\lambda\rangle \\
&\ge \langle \lambda-\rho_{n_i}, B\lambda - \frac{\rho_{n_i}-z_{n_i}}{s_{n_i}} - Bz_{n_i}\rangle \\
&= \langle \lambda-\rho_{n_i}, B\lambda - B\rho_{n_i}\rangle + \langle \lambda-\rho_{n_i}, B\rho_{n_i} - Bz_{n_i}\rangle \\
&\quad - \langle \lambda-\rho_{n_i}, \frac{\rho_{n_i}-z_{n_i}}{s_{n_i}}\rangle \\
&\ge \langle \lambda-\rho_{n_i}, B\rho_{n_i} - Bz_{n_i}\rangle - \langle \lambda-\rho_{n_i}, \frac{\rho_{n_i}-z_{n_i}}{s_{n_i}}\rangle,
\end{aligned}$$

which implies that $\langle \lambda-\xi, \lambda'\rangle \ge 0$. We have $\xi \in S^{-1}0$ and hence $\xi \in VI(C,B)$. Since $\xi = P_\Omega(\gamma f + (I-M))(\xi)$, we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle \gamma f(q) - Mq, x_n - q\rangle &= \lim_{n\to\infty}\langle \gamma f(q) - Mq, x_{n_i} - q\rangle \\
&= \langle \gamma f(q) - Mq, \xi - q\rangle \le 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|y_n - q\|^2 &\le (1-\beta_n\bar{\gamma})^2\|\rho_n - q\|^2 + 2\beta_n\gamma\langle f(z_n) - f(q), y_n - q\rangle \\
&\quad + 2\beta_n\langle \gamma f(q) - Aq, y_n - q\rangle \\
&\le (1-\beta_n\bar{\gamma})^2\|x_n - q\|^2 + 2\beta_n\gamma\alpha\|z_n - q\|\|y_n - q\| \\
&\quad + 2\beta_n\langle \gamma f(q) - Aq, y_n - q\rangle \\
&\le (1-\beta_n\bar{\gamma})^2\|x_n - q\|^2 + \beta_n\gamma\alpha(\|z_n - q\|^2 + \|y_n - q\|^2) \\
&\quad + 2\beta_n\langle \gamma f(q) - Aq, y_n - q\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|y_n - q\|^2 &\le \frac{(1-\beta_n\bar{\gamma})^2 + \beta_n\gamma\alpha}{1-\beta_n\gamma\alpha}\|x_n - q\|^2 + \frac{2\beta_n}{1-\beta_n\gamma\alpha}\langle \gamma f(q) - Aq, y_n - q\rangle \\
&= \frac{(1-2\beta_n\bar{\gamma}+\beta_n\alpha\gamma)}{1-\beta_n\gamma\alpha}\|x_n - q\|^2 + \frac{\beta_n^2\bar{\gamma}^2}{1-\beta_n\gamma\alpha}\|x_n - q\|^2 \\
&\quad + \frac{2\beta_n}{1-\beta_n\gamma\alpha}\langle \gamma f(q) - Aq, y_n - q\rangle \\
&\le [1 - \frac{2\beta_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\gamma\alpha}]\|x_n - q\|^2 \\
&\quad + \frac{2\beta_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\gamma\alpha}[\frac{1}{\bar{\gamma}-\alpha\gamma}\langle \gamma f(q) - Aq, y_n - q\rangle + \frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}M].
\end{aligned} \tag{3.9}$$

On the other hand, we have $\|x_{n+1} - p\|^2 \le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2$.

From (3.9), we find that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le [1 - (1-\alpha_n)\frac{2\beta_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\gamma\alpha}]\|x_n - q\|^2 \\
&\quad + (1-\alpha_n)\frac{2\beta_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\gamma\alpha}[\frac{1}{\bar{\gamma}-\alpha\gamma}\langle \gamma f(q) - Aq, y_n - q\rangle + \frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}R_2],
\end{aligned} \tag{3.10}$$

where R_2 is an appropriate constant. Put $l_n = (1-\alpha_n)\frac{2\beta_n(\bar{\gamma}-\alpha_n\gamma)}{1-\beta_n\alpha\gamma}$ and $t_n = \frac{1}{\bar{\gamma}-\alpha\gamma}\langle \gamma f(q) - Aq, x_{n+1} - q\rangle + \frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}R_2$. That is, $\|x_{n+1} - q\|^2 \le (1-l_n)\|x_n - q\|^2 + l_n t_n$. It follows that

$$\lim_{n\to\infty} l_n = 0, \sum_{n=1}^{\infty} l_n = \infty \text{ and } \limsup_{n\to\infty} t_n \le 0.$$

Apply Lemma 2.1 to conclude $x_n \to q$. This completes the proof.

From Theorem 3.1, we have the following results immediately.

Corollary 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping. Let f be a contraction of H into itself with a coefficient κ $(0 < \kappa < 1)$ and let M be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ with $0 < \gamma < \bar{\gamma}/\kappa$. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1), (A2), (A3) and (A4). Assume that $GEP(F,A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases}
r_n F(z_n, \eta) + r_n\langle Ax_n, \eta - z_n\rangle + \langle \eta - z_n, z_n - x_n\rangle \ge 0, \forall \eta \in C, \\
x_{n+1} = \beta_n(1-\alpha_n)\gamma f(z_n) + (1-\alpha_n)(I - \beta_n M)z_n + \alpha_n x_n, \quad n \ge 1,
\end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *are two sequence in* $(0,1)$ *and* $\{r_n\} \subset (0,\infty)$. *Assume that the control sequences satisfy the following conditions:* $\lim_{n\to\infty}\beta_n = 0, \sum_{n=1}^{\infty}\beta_n = \infty$, $\lim_{n\to\infty}|r_{n+1}-r_n| = 0$, $0 < a < \alpha_n < b < 1$, *and* $0 < e \leq r_n \leq f < 2\beta$, *where* a,b,e,f *are two constants. Then,* $\{x_n\}$ *converges strongly to* $q \in GEP(F,A)$, *where* $q = P_{GEP(F,A)}(\gamma f + (I-M))(q)$, *which solves the following variational inequality* $\langle \gamma f(q) - Mq, p-q\rangle \leq 0, \ \forall p \in GEP(F,A)$.

Corollary 3.3. *Let* C *be a nonempty closed convex subset of a Hilbert space* H. *Let* $B : C \to H$ *be a* β*-inverse-strongly monotone mapping. Let* f *be a contraction of* H *into itself with a coefficient* κ $(0 < \kappa < 1)$ *and let* M *be a strongly positive linear bounded operator with coefficient* $\bar{\gamma} > 0$ *with* $0 < \gamma < \bar{\gamma}/\kappa$. *Assume that* $VI(C,B) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $x_1 \in C$ *and*

$$x_{n+1} = \beta_n(1-\alpha_n)\gamma f(z_n) + (1-\alpha_n)(I-\beta_n M)P_C(x_n - s_n Bx_n) + \alpha_n x_n, \quad n \geq 1,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *are two sequence in* $(0,1)$ *and* $\{s_n\} \subset (0,\infty)$. *Assume that the control sequences satisfy the following conditions:* $\lim_{n\to\infty}\beta_n = 0, \sum_{n=1}^{\infty}\beta_n = \infty$, $\lim_{n\to\infty}|s_{n+1}-s_n| = 0$, $0 < a < \alpha_n < b < 1$, $0 < c \leq s_n \leq d < 2\alpha$, *where* a,b,c *and* d *are constants. Then,* $\{x_n\}$ *converges strongly to* $q \in VI(C,B)$, *where* $q = P_{VI(C,B)}(\gamma f + (I-M))(q)$, *which solves the following variational inequality* $\langle \gamma f(q) - Mq, p-q\rangle \leq 0, \ \forall p \in VI(C,B)$.

Conclusion In this paper, we studied monotone variational inequality and generalized equilibrium problems and constructed a general viscosity iterative algorithm for solving them. It deserves mentioning there is no projection involved in the algorithm. Our convergence analysis ensures that the proposed algorithm converges in norm to a special zero that is also a unique solution to a monotone variational inequality under mild conditions.

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