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SPLITTING ALGORITHMS FOR COMMON SOLUTIONS OF NONLINEAR PROBLEMS

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Abstract. The aim of this paper is to study common solution problems of two nonlinear problems. A weak convergence theorem is obtained in a Banach space. The results improve and extend the corresponding results announced recently.

Keywords. Accretive operator; Monotone operator; Variational inequality; Projection; Convergence.

1. Introduction-Preliminaries

Let E be a real Banach space with the dual E^*. Recall the following generalized duality map $\mathfrak{J}_q(x) : E \to 2^{E^*}$, where $q > 1$, defined as

$$\mathfrak{J}_q(x) := \{x^* \in E^* : \langle x^*, x\rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}, \quad \forall x \in E.$$

Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{f^* \in E^* : \|x\|^2 = \langle x, f^*\rangle = \|f^*\|^2\}.$$

Let $U_E = \{x \in E : \|x\| = 1\}$. Recall that a Banach space E is said to be strictly convex if and only if $\|x+y\| < 2$ for all $x, y \in U_E$ with $x \neq y$. E is said to be uniformly convex if and only if $\lim_{n\to\infty} \|u_n - v_n\| = 0$, where $\{u_n\}$ and $\{v_n\}$ in U_E and $\lim_{n\to\infty} \|u_n + v_n\| = 2$.

Let $\rho_E : [0,\infty) \to [0,\infty)$ be the modulus of smoothness of E by

$$\rho_E(t) = \sup\{\frac{\|x+y\| - \|x-y\|}{2} - 1 : x \in U_E, \|y\| \le t\}.$$

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A Banach space E is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. Let $q > 1$. E is said to be q-uniformly smooth if there exists a fixed constant $c > 0$ such that $\rho_E(t) \leq ct^q$. It is known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth, and hence the norm of E is uniformly Fréchet differentiable, in particular, the norm of E is Fréchet differentiable.

Let T be a mapping on E. The fixed point set of T is denoted by $F(T)$. Recall that T is said to be nonexpansive iff

$$\|Tx - Ty\| \leq \|x, y\|, \quad \forall x, y \in C.$$

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \cup\{Az : z \in D(A)\}$ is said to be accretive if, for $t > 0$ and $x, y \in D(A)$,

$$\|x - y\| \leq \|x - y + t(u - v)\|, \quad \forall u \in Ax, v \in Ay.$$

It follows from Kato [1] that A is accretive if and only if, for $x, y \in D(A)$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that

$$\langle u - v, j(x - y)\rangle \geq 0.$$

An accretive operator A is said to be m-accretive if $R(I + rA) = E$ for all $r > 0$. In a real Hilbert space, an operator A is m-accretive if and only if A is maximal monotone. For an accretive operator A, we can define a nonexpansive single valued mapping $J_r^A : R(I + rA) \to D(A)$ by $J_r^A = (I + rA)^{-1}$ for each $r > 0$, which is called the resolvent of A.

Recall that a single valued operator $A : E \to E$ is said to be α-inverse strongly accretive if there exists a constant $\alpha > 0$ and some $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, J(x - y)\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in E.$$

Recently, zero point problems of accretive operators have been extensively investigated via fixed point methods; see [2-13] and the references therein. In this paper, we investigate the zero point problem of the sum of two accretive operators based on a splitting methods. A weak convergence theorem is obtained in a Banach space. The results improve and extend the corresponding results announced recently. In order to obtain the main results of this paper, we need the following tools.

Lemma 1.1. [14] *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle$$

and

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + 2\|Ky\|^2, \quad \forall x, y \in E.$$

Lemma 1.2. *Let E be a real Banach space and let C be a nonempty closed and convex subset of E. Let $A : C \to E$ be a single valued operator and let $B : E \to 2^E$ be an m-accretive operator. Then*

$$F(J_a^B(I - aA)) = (A+B)^{-1}(0),$$

where J_a^B is the resolvent of B for $a > 0$.

Lemma 1.3. [14] *Let $p > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that*

$$\|ax + (1-a)y\|^p \leq a\|x\|^p + (1-a)\|y\|^p - \big(a^p(1-a) + (1-a)^p a\big)\varphi(\|x-y\|),$$

for all $x, y \in B_r(0) := \{x \in E : \|x\| \leq r\}$ and $a \in [0,1]$.

Lemma 1.4. [15] *Let E be a real uniformly convex Banach space and let C be a nonempty closed convex and bounded subset of E. Then there is a strictly increasing and continuous convex function $\psi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that, for every Lipschitzian continuous mapping $T : C \to C$ and, for all $x, y \in C$ and $t \in [0,1]$, the following inequality holds:*

$$\|T(tx + (1-t)y) - (tTx + (1-t)Ty)\| \leq L\psi^{-1}\big(\|x-y\| - L^{-1}\|Tx - Ty\|\big),$$

where $L \geq 1$ is the Lipschitz constant of T.

Lemma 1.5. [16] *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec-Klee property. Suppose that $\{x_n\}$ is a bounded sequence such that $\lim_{n\to\infty}\|ax_n + (1-a)p_1 - p_2\|$ exists for all $a \in [0,1]$ and $p_1, p_2 \in \omega_w(x_n)$, where $\omega_w(x_n) : \{x : \exists x_{n_i} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$ Then $\omega_w(x_n)$ is a singleton.*

Lemma 1.6. [17] *Let E be a real uniformly convex Banach space, C a nonempty closed, and convex subset of E and $T : C \to C$ a nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

2. Main results

Theorem 2.1. *Let E be a real uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K and let C be a closed convex subset of E. Let $A : C \to E$ be an α-inverse strongly accretive operator and let $B : Dom(B) \subset E \to 2^E$ be an m-accretive operator such that $Dom(B) \subset C$. Assume $(A+B)^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner: $x_0 \in C$ and*

$$x_{n+1} = (1-\alpha_n)(I + r_n B)^{-1}(x_n - r_n A x_n) + \alpha_n x_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ and $\{r_n\}$ are real sequences satisfying the following restrictions: $0 \leq \alpha_n \leq \alpha < 1$ and $0 < r \leq r_n \leq r' < \frac{\alpha}{K^2}$. Then $\{x_n\}$ converges weakly to some zero of $A+B$.

Proof. From Lemma 1.1, one has

$$\begin{aligned} &\|(I - r_n A)x - (I - r_n A)y\|^2 \\ &\leq \|x-y\|^2 - 2r_n\langle Ax - Ay, J(x-y)\rangle + 2K^2 r_n^2 \|Ax - Ay\|^2 \\ &\leq \|x-y\|^2 - 2r_n\alpha\|Ax-Ay\|^q + 2K^2 r_n^2\|Ax-Ay\|^2 \\ &= \|x-y\|^2 - 2r_n(\alpha - K^2 r_n\|Ax-Ay\|^2. \end{aligned} \tag{3.1}$$

From the restriction on $\{r_n\}$, one sees that

$$\|(I - r_n A)x - (I - r_n A)y\| \leq \|x-y\|. \tag{3.2}$$

Fixing $p \in (A+B)^{-1}(0)$, one has

$$\begin{aligned} \|x_n - p\| &\geq \alpha_n\|x_n - p\| + (1-\alpha_n)\|(x_n - r_n A x_n) - (p - r_n A)p\| \\ &\geq \alpha_n\|x_n - p\| + (1-\alpha_n)\|J_{r_n}(x_n - r_n A x_n) - p\| \\ &\geq \|x_{n+1} - p\|. \end{aligned}$$

This shows that $\lim_{n\to\infty}\|x_n - p\|$ exists, in particular, $\{x_n\}$ is bounded. Putting $y_n = J_{r_n}(x_n - r_nAx_n)$, we find from Lemma 1.3 that

$$\begin{aligned}
&\|(I-r_nA)x_n-(I-r_nA)p\|^2-\frac{1}{4}\varphi\Big(\|(y_n-p)-\big((I-r_nA)x_n-(I-r_nA)p\big)\|\Big)\\
&\geq \frac{1}{2}\|y_n-p\|^2+\frac{1}{2}\|(I-r_nA)x_n-(I-r_nA)p\|^2\\
&\quad-\frac{1}{4}\varphi\Big(\|(y_n-p)-\big((I-r_nA)x_n-(I-r_nA)p\big)\|\Big)\\
&\geq \Big\|\frac{1}{2}(y_n-p)+\frac{1}{2}\big((I-r_nA)x_n-(I-r_nA)p\big)\Big\|^2.
\end{aligned}\tag{3.3}$$

Substituting (3.1) into (3.3), one finds that

$$\begin{aligned}
&\Big\|\frac{1}{2}(y_n-p)+\frac{1}{2}\big((I-r_nA)x_n-(I-r_nA)p\big)\Big\|^2\\
&\leq \|x_n-p\|^2-2r_n(\alpha-K^2r_n\|Ax-Ay\|^2\\
&\quad-\frac{1}{4}\varphi\Big(\|(y_n-p)-\big((I-r_nA)x_n-(I-r_nA)p\big)\|\Big).
\end{aligned}\tag{3.4}$$

In view of the acctiveness of B, we find that

$$\begin{aligned}
&\Big\|\frac{1}{2}\big((I-r_nA)x_n-(I-r_nA)p\big)+\frac{1}{2}(y_n-p)\Big\|\\
&=\Big\|\frac{r_n}{2}\Big(\frac{x_n-r_nAx_n-y_n}{r_n}-\frac{(I-r_nA)p-p}{r_n}\Big)+y_n-p\Big\|\\
&\geq \|y_n-p\|
\end{aligned}\tag{3.5}$$

Combining (3.4) with (3.5), we see that

$$\begin{aligned}
&\|x_n-p\|^2-2r_n(\alpha-K^2r_n\|Ax-Ay\|^2\\
&\quad-\frac{1}{4}\varphi\Big(\|(y_n-p)-\big((I-r_nA)x_n-(I-r_nA)p\big)\|\Big)\\
&\geq \|y_n-p\|^2
\end{aligned}\tag{3.6}$$

It follows that

$$\begin{aligned}
&\|x_n-p\|^2-2r_n(\alpha-K^2r_n\|Ax-Ay\|^2\\
&\quad-(1-\alpha_n)\frac{1}{4}\varphi\Big(\|(y_n-p)-\big((I-r_nA)x_n-(I-r_nA)p\big)\|\Big)\\
&\geq \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2\\
&\geq \|x_{n+1}-p\|^2.
\end{aligned}$$

Hence, we have

$$\|x_n-p\|^2-2r_n(\alpha-K^2r_n\|Ax-Ay\|^2-\|x_{n+1}-p\|^2$$
$$\geq(1-\alpha_n)\frac{1}{4}\varphi\Big(\|(y_n-p)-\big((I-r_nA)x_n-(I-r_nA)p\big)\|\Big)$$

and

$$\|x_n-p\|^2-\|x_{n+1}-p\|^2-(1-\alpha_n)\frac{1}{4}\varphi\Big(\|(y_n-p)-\big((I-r_nA)x_n-(I-r_nA)p\big)\|\Big)$$
$$\geq 2r_n(\alpha-K^2r_n\|Ax-Ay\|^2.$$

In view of $0\leq\alpha_n\leq\alpha<1$ and $0<r\leq r_n\leq r'<\frac{\alpha}{K^2}$, one sees that

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0 \tag{3.7}$$

and

$$\lim_{n\to\infty}\|y_n-x_n+r_nAx_n-r_nAp\|=0. \tag{3.8}$$

Since $\|y_n-x_n\|\leq\|y_n-x_n+r_nAx_n-r_nAp\|+r_n\|Ax_n-Ap\|$, we find that

$$\lim_{n\to\infty}\|J_{r_n}(x_n-r_nAx_n)-x_n\|=0. \tag{3.9}$$

Notice that

$$0\leq\Big\langle\frac{x_n-J_r(I-rA)x_n}{r}-\frac{x_n-J_{r_n}(I-r_nA)x_n}{r_n},J\big(J_r(I-rA)x_n-J_{r_n}(I-r_nA)x_n\big)\Big\rangle.$$

Hence, we find that

$$\|x_n-J_{r_n}(I-r_nA)x_n\|\|J_r(I-rA)x_n-J_{r_n}(I-r_nA)x_n\|$$
$$\geq\frac{r_n-r}{r_n}\langle x_n-J_{r_n}(I-r_nA)x_n,J\big(J_r(I-rA)x_n-J_{r_n}(I-r_nA)x_n\big)\rangle$$
$$\geq\|J_r(I-rA)x_n-J_{r_n}(I-r_nA)x_n\|^2.$$

This implies that $\|x_n-J_{r_n}(I-r_nA)x_n\|\geq\|J_r(I-rA)x_n-J_{r_n}(I-r_nA)y_n\|$. It follows that

$$\|J_r(I-rA)x_n-x_n\|\leq\|J_r(I-rA)x_n-J_{r_n}(I-r_nA)x_n\|$$
$$+\|J_{r_n}(I-r_nA)x_n-x_n\|$$
$$\leq 2\|J_{r_n}(I-r_nA)x_n-x_n\|.$$

From (3.9), we arrive at

$$\lim_{n\to\infty}\|J_r(x_n-rAx_n)-x_n\|=0. \tag{3.10}$$

Define mappings $T_n : C \to C$ by

$$T_n x := \alpha_n x + (1-\alpha_n) J_{r_n}((I - r_n A)x), \quad \forall x \in C.$$

Set

$$T_{n+m-1} T_{n+m-2} \cdots T_n = S_{n,m}, \quad \forall n, m \geq 1.$$

Then $S_{n,m} x_n = x_{n+m}$ and $S_{n,m}$ is nonexpansive. For all $t \in [0,1]$ and $n, m \geq 1$, put

$$a_n(t) = \|t x_n + (1-t) p_1 - p_2\|,$$

and

$$b_{n,m} = \|S_{n,m}(t x_n + (1-t) p_1) - (t x_{n+m} + (1-t) p_1)\|,$$

where p_1 and p_2 are zeros of $A + B$. Using Lemma 1.4, we find that

$$\begin{aligned} b_{n,m} &\leq \psi^{-1}\big(\|x_n - p_1\| - \|S_{n,m} x_n - S_{n,m} p_1\|\big) \\ &= \psi^{-1}\big(\|x_n - p_1\| - \|x_{n+m} - p_1 + p_1 - S_{n,m} p_1\|\big) \\ &\leq \psi^{-1}\big(\|x_n - p_1\| - (\|x_{n+m} - p_1\| - \|p_1 - S_{n,m} p_1\|)\big) \\ &\leq \psi^{-1}\big(\|x_n - p_1\| - \|x_{n+m} - p_1\|\big). \end{aligned}$$

It follows that $\{b_{n,m}\}$ converges uniformly to zero as $n \to \infty$ for all $m \geq 1$. Hence,

$$\begin{aligned} a_{n+m}(t) &\leq b_{n,m} + \|S_{n,m}(t x_n + (1-t) p_1) - p_2\| \\ &\leq b_{n,m} + \|S_{n,m}(t x_n + (1-t) p_1) - S_{n,m} p_2\| + \|S_{n,m} p_2 - p_2\| \\ &\leq b_{n,m} + a_n(t) + \|S_{n,m} p_2 - p_2\| \\ &\leq b_{n,m} + a_n(t). \end{aligned}$$

Taking lim sup as $m \to \infty$ and then the lim inf as $n \to \infty$, we find that $\lim_{n\to\infty} a_n(t)$ for any $t \in [0,1]$. In view of Lemma 1.5, we see that $\omega_w(x_n) \subset (A+B)^{-1}(0)$. This implies from Lemma 1.6 that $\omega_w(x_n)$ is just one point. This proves the proof.

From Theorem 3.1, we immediately have the following results.

Corollary 2.2. *Let H be a real Hilbert space and let C be a closed convex subset of H. Let $A : C \to E$ be an α-inverse strongly monotone operator and let $B : Dom(B) \subset H \to 2^H$ be an* *m-accretive operator such that* $Dom(B) \subset C$*. Assume* $(A+B)^{-1}(0) \neq \emptyset$*. Let* $\{x_n\}$ *be a sequence generated in the following manner:* $x_0 \in C$ *and*

$$x_{n+1} = (1-\alpha_n)(I+r_nB)^{-1}(x_n - r_nAx_n) + \alpha_n x_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ *and* $\{r_n\}$ *are real sequences satisfying the following restrictions:* $0 \leq \alpha_n \leq \alpha < 1$ *and* $0 < r \leq r_n \leq r' < 2\alpha$*. Then* $\{x_n\}$ *converges weakly to some zero of* $A+B$*.*

Corollary 2.3. *Let* H *be a real Hilbert space and let* C *be a closed convex subset of* H*. Let* $A : C \to E$ *be an* α*-inverse strongly monotone operator and let* $Proj_C$ *be the metric projection from* H *onto* C*. Assume* $VI(C,A) \neq \emptyset$*. Let* $\{x_n\}$ *be a sequence generated in the following manner:* $x_0 \in C$ *and*

$$x_{n+1} = (1-\alpha_n)Proj_C(x_n - r_nAx_n) + \alpha_n x_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ *and* $\{r_n\}$ *are real sequences satisfying the following restrictions:* $0 \leq \alpha_n \leq \alpha < 1$ *and* $0 < r \leq r_n \leq r' < 2\alpha$*. Then* $\{x_n\}$ *converges weakly to some zero of* $VI(C,A)$*.*

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