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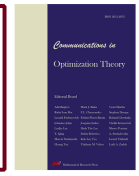

PARTIAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS AND OPTIMAL CONTROLS IN BANACH SPACES

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Abstract. In this work, we study of the solvability and the existence of optimal controls of some partial functional integrodifferential equations with classical Cauchy initial condition in Banach spaces. We assume that the linear part admits a resolvent operator in the sense of Grimmer. Firstly, we investigate the existence and uniqueness of mild solutions. Secondly, we prove the existence of optimal controls for the integrodifferential equation. An example is also given to illustrate the main results.

Keywords: Partial functional integrodifferential equation; Resolvent operator; Solvability; Mild solution; Optimal control.

1. Introduction

The aim of this work is to study the existence of mild solutions and the optimal controls of some systems that arise in the analysis of heat conduction in materials with memory [6], and viscoelasticity and take the form of the following partial functional integrodifferential equation in a Banach space $(X, \|\cdot\|)$:

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$$
\begin{cases} x'(t) = Ax(t) \; + \; \int_0^t B(t-s)x(s)ds \; + \; f(t,x(t)) + C(t)u(t) \; \text{ for } t \in I = [0,b] \\ \\ x(0) = x_0 \in X, \end{cases} \tag{1.1}
$$

where $f : I \times X \to X$ is a function satisfying some conditions; $A : \mathscr{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\left(T(t)\right)_{t\geq 0}$ on a separable reflexive Banach space X; for $t \geq 0$, $B(t)$ is a closed linear operator with domain $\mathscr{D}(B(t)) \supset \mathscr{D}(A)$. The control $u(t)$ takes values from another separable reflexive Banach space U. The operator $C(t)$ belongs to $\mathscr{L}(U,X)$, the Banach space of bounded linear operators from U into X.

In many areas of applications such as engineering, electronics, fluid dynamics, physical sciences, etc..., integrodifferential equations appear and have received considerable attention during the last decades. In [6], R. Grimmer proved the existence and uniqueness of resolvent operators for these integrodifferential equations that give the variation of parameter formula for the solution. In recent years, much work has been done on the existence and regularity of solutions of nonlinear integrodifferential equations with various initial conditions by many authors by applying the resolvent operator theory, for integral equations see e.g., [4] and the references therein. Problems of controllability and existence of optimal controls for nonlinear differential equations have been studied extensively by many authors under various hypotheses (see e.g., [2], [5],[10],[11],[16] [17]), but little is known and done about the existence of optimal controls for integrodifferential equations using the resolvent operator theory. Wang and Zhou [21] discussed the optimal controls of a Lagrange problem for the following fractional evolution equations:

$$
\begin{cases} D^q x(t) = -Ax(t) \; + f(t,x(t)) + C(t)u(t) \; \text{ for } t \in [0,b] \\ \\ x(0) = x_0 \in X, \end{cases}
$$

where D^q denotes the Caputo fractional derivative of order $q \in (0,1)$ and $-A : \mathscr{D}(A) \to X$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators. In [12], the authors studied the existence of mild solutions and the optimal controls of a Lagrange problem for the following impulsive fractional semilinear differential equations,

$$
\begin{cases} {}^C D_t^\alpha x(t) = Ax(t) \; + f\left(t, x(t), \int_0^t g(t,s,x(s))ds\right) + C(t)u(t) \; \text{ for } t \in [0,b], \; t \neq t_k \\ \\ \Delta x(t_k) = I_k(x(t_k^-)), \; k = 1,2,\cdots m \\ \\ x(0) = x_0 \in X, \end{cases}
$$

where ${}^C D_t^\alpha$ denotes the Caputo fractional derivative of order $\alpha \in (0,1]$ with lower limit zero and $A : \mathscr{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup. They used the techniques of *a priori* estimation. In [22], the authors studied the optimal controls for nonlinear impulsive integrodifferential equations of mixed type on Banach spaces. Motivated by these works, we investigate the solvability and the existence of optimal controls of a Lagrange problem for equation (1.1), using the techniques of *a priori* estimation of mild solutions. The existence and uniqueness of mild solutions is obtained using the theory of resolvent operator for integral equations. Furthermore, to the best of our knowledge, the optimal controls for partial functional integrodifferential equation (1.1) with classical Cauchy initial conditions are untreated in the literature, and this fact motivates us to extend the existing ones and make new development of the present work on this issue.

As a motivation for the problem studied in this work, we consider a heat flow in a rigid body Ω of a material with memory. Let $w(t,\xi),\ e(t,\xi),\ q(t,\xi)$ and $s(t,\xi)$ denote respectively the temperature, the internal energy, the heat flux, and the external heat supply at time t and position ξ. The balance law for the heat transfer is given by:

$$e_t(t,\xi)+\operatorname{div} q(t,\xi)=s(t,\xi) \tag{1.2}$$

and the physical properties of the body suggest the dependence of e and q on w and ∇w, respectively. For instance assuming the Fourier Law i.e.,

$$e(t,\xi)=c_1 w(t,\xi) \tag{1.3}$$

$$q(t,\xi)=-c_2\nabla w(t,\xi), \tag{1.4}$$

where c_1, c_2 are positive constants, one deduces from (1.2) the classical heat equation

$$w_t(t,\xi)=c\Delta w(t,\xi)+f(t,\xi) \tag{1.5}$$

with $c=c_1^{-1}c_2$ and $f(t,\xi)=c_1^{-1}s(t,\xi)$. In many materials the assumptions (1.3), (1.4) are not justified because they take no account of the memory effects: several models have been proposed to overcome this difficulty, see e.g. [3, 9, 18]: one of them consists in substituting (1.4) with

$$q(t,\xi)=-c_2\nabla w(t,\xi)-\int_{-\infty}^{t}h(t-s)\nabla w(s,\xi)ds. \tag{1.6}$$

Taking for simplicity $c_1=c_2=1$, we get from (1.2), (1.3) and (1.6)

$$w_t(t,\xi)=\Delta w(t,\xi)+\int_{-\infty}^{t}h(t-s)\Delta w(s,\xi)ds+s(t,\xi). \tag{1.7}$$

If we assume that the thermal history w of the body Ω is known up to $t=0$, the temperature of the boundary $\partial\Omega$ of Ω is constant (=0) for all t, and the external heat flux depends on the this thermal history of the body, we are led to the following system:

$$
\left\{
\begin{array}{l}
w_t(t,\xi) = \Delta w(t,\xi) + \displaystyle\int_0^t h(t-s)\Delta w(s,\xi)ds + f(t,w(t,\xi)) \ \text{ for } (t,\xi) \in [0,b]\times\Omega \\[2em]
w(0,\xi) = w_0(\xi), \ \ \xi \in \overline{\Omega}
\end{array}
\right. \tag{1.8}
$$

where $b > 0$ is arbitrarily fixed. If we prescribe h (in addition to f and w_0), then (1.8) is a Cauchy-Dirichlet problem for an integrodifferential equation in the unknown w, which has been studied by several authors in the last decades, see e.g., [7, 8, 15] and references therein.

Now define

$$x(t)(\xi) \;=\; w(t,\xi)$$

$$Ax \;=\; \Delta x$$

$$f(t,x(t))(\xi) \;=\; f(t,w(t,(\xi)))(\xi) \quad \text{for } t \in [0,b] \text{ and } \xi \in \Omega$$

$$(B(t)x)(\xi) \;=\; h(t)\Delta x(t)(\xi) \ \text{ for } \ t \in [0,b], \text{ and } \ \xi \in \Omega.$$

Then, equation (1.8) can be transformed into the following abstract form:

$$
\left\{
\begin{array}{l}
x'(t) = Ax(t) \ + \ \displaystyle\int_0^t B(t-s)x(s)ds \ + \ f(t,x(t)) \ \text{ for } t \in I = [0,b], \\[2em]
x(0) = x_0 \in X,
\end{array}
\right. \tag{1.9}
$$

where X is a Banach space. Equation (1.9) has been studied by many authors (see e.g., [13] and the references contained in it). But to the best of our knowledge, this equation has never been considered for optimal control.

The rest of the paper is organized as follows: In section 2, we present some basic definitions and preliminaries results, which will be used in the subsequent sections. In section 3, we obtain an *a priori* estimation of mild solutions of equation (1.1). In section 4, sufficient conditions are established for the existence and uniqueness of mild solutions of equation (1.1), by applying a well known fixed point theorem, and extension by continuity techniques. In section 5, we investigate the existence of optimal controls of a Lagrange optimal control problem for equation (1.1). Finally, in section 6, an example is given to illustrate the results.

2. Resolvent Operators and Balder's Theorem

In this section we introduce some definitions and Lemmas that will be used throughout the paper.

A measurable function $x : I \to X$ is Bochner integrable if and only if $\|x\|$ is Lebesgue integrable. We denote by

$L^1_B(I,X)$ the Banach space of functions $x : I \to X$ which are Bochner integrable normed by

$$\|x\| = \int_0^b \|x(t)\| dt.$$

Consider the following linear homogeneous equation:

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t B(t-s)x(s)ds \quad \text{for} \quad t \geq 0 \\ \\ x(0) = x_0 \in X. \end{cases} \tag{2.1}$$

where A and $B(t)$ are closed linear operators on a Banach space X.
In the sequel, we assume A and $\big(B(t)\big)_{t\geq 0}$ satisfy the following conditions:

($\mathbf{H_1}$) A is a densely defined closed linear operator in X. Hence $\mathscr{D}(A)$ is a Banach space equipped with the graph norm defined by, $|y| = \|Ay\| + \|y\|$ which will be denoted by $(X_1, |\cdot|)$.

($\mathbf{H_2}$) $\big(B(t)\big)_{t\geq 0}$ is a family of linear operators on X such that $B(t)$ is continuous when regarded as a linear map from $(X_1, |\cdot|)$ into $(X, \|\cdot\|)$ for almost all $t \geq 0$ and the map $t \mapsto B(t)y$ is measurable for all $y \in X_1$ and $t \geq 0$, and belongs to $W^{1,1}(\mathbb{R}^+, X)$. Moreover there is a locally integrable function $b : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|B(t)y\| \ \leq \ b(t)|y| \quad \text{and} \quad \left\| \frac{d}{dt} B(t)y \right\| \ \leq \ b(t)|y| \ .$$

Remark 1. Note that ($\mathbf{H_2}$) is satisfied in the modelling of Heat Conduction in materials with memory and viscosity. More details can be found in [13].

Let $\mathscr{L}(X)$ be the Banach space of bounded linear operators on X,

Definition 2.1. [4] A resolvent operator $\big(R(t)\big)_{t\geq 0}$ for equation (2.1) is a bounded operator valued function

$$R : \ [0, +\infty) \ \longrightarrow \ \mathscr{L}(X)$$

such that

(i) $R(0) = Id_X$ and $\|R(t)\| \leq Ne^{\beta t}$ for some constants N and β.

(ii) For all $x \in X$, the map $t \mapsto R(t)x$ is continuous for $t \geq 0$.

(iii) Moreover for $x \in X_1$, $R(\cdot)x \in \mathscr{C}^1(\mathbb{R}^+;X) \cap \mathscr{C}(\mathbb{R}^+;X_1)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds. \end{aligned}$$

Observe that the map defined on $\mathbb{R}^+$ by $t \mapsto R(t)x_0$ solves equation (2.1) for $x_0 \in \mathscr{D}(A)$.

Theorem 2.2. [6] *Assume that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. Then, the linear equation (2.1) has a unique resolvent operator* $\big(R(t)\big)_{t\geq 0}$.

Remark 2. In general, the resolvent operator $\big(R(t)\big)_{t\geq 0}$ for equation (2.1) does not satisfy the semigroup law, namely,

$$R(t+s) \neq R(t)R(s) \qquad \text{for some } t,s>0.$$

The following Theorem is needed in the proof of the existence of optimal controls.

Theorem 2.3. *(Balder's Theorem,* [1]*) Let* $(\Sigma,\mathscr{F},\mu)$ *be a finite nonatomic measure space,* $(Y,\|\cdot\|)$ *a separable Banach space, and* $(V,|\cdot|)$ *a separable reflexive Banach space, and* V' *its dual. Let* $\theta : \Sigma \times Y \times V \to (-\infty,+\infty]$ *be a given* $\mathscr{F} \times \mathscr{L}(Y \times V)$*- measurable function. The associated integral functional* $I_\theta : L^1_Y \times L^1_V \to [-\infty,+\infty]$ *is defined by:*

$$I_\theta(x,v) = \int_\Sigma \theta(t,x(t),v(t))\mu(dt),$$

where L^1_Y *denotes the space of all absolutely summable functions from* Σ *to* Y.
The following three conditions

(i) $\theta(t,\cdot,\cdot)$ *is sequencially lower semicontinuous on* $X \times V$, μ*-a.e.,*

(ii) $\theta(t,x,\cdot)$ *is convex on* V *for* $x \in Y$, μ*-a.e.,*

(iii) *there exist* $\sigma > 0$ *and* $\psi \in L^1_{\mathbb{R}}$ *such that*

$$\theta(t,x,v) \geq \psi(t) - \sigma(\|x\| + |v|) \text{ for all } x \in Y,\ v \in V,\ \mu\text{-a.e.},$$

are sufficient for sequential strong-weak lower semicontinuity I_θ *on* $L^1_Y \times L^1_V$*. Moreover, they are also necessary, provided that* $I_\theta(\bar{x},\bar{v}) < +\infty$ *for some* $\bar{x} \in L^1_Y, \bar{v} \in L^1_V$.

Theorem 2.4. *(Mazur's Lemma,* [14]*) Let Z be a Banach space and G be a convex and closed set in Z. Then G is weakly closed in Z.*

3. Existence of mild solutions for equation (1.1)

We make the following assumptions.

$(\mathbf{H_3})$ The function $f : I \times X \to X$ satisfies the following conditions:

(i) $f(\cdot, x)$ is measurable for $x \in X$,

(ii) for any $\rho > 0$, there exists $L_f(\rho) > 0$ such that

$$\|f(t,x) - f(t,y)\| \leq L_f(\rho)\|x - y\| \quad \text{for } \|x\| \leq \rho,\ \|y\| \leq \rho \text{ and } t \in [0,b],$$

(iii) there exists $a_f > 0$ such that

$$\|f(t,x\| \leq a_f(1 + \|x\|) \quad \text{for all } \ x \in X \ \text{ and } \quad t \in [0,b].$$

$(\mathbf{H_4})$ Let U be the separable reflexive Banach space from which the control u takes values and assume $C \in L^\infty(I; \mathscr{L}(U,X))$.

$(\mathbf{H_5})$ The multivalued map $\Gamma : I \to 2^U \setminus \{\emptyset\}$ has closed, convex, and bounded values, Γ is graph measurable, and $\Gamma(\cdot) \subseteq \Omega$ where Ω is a bounded set in U.

We denote by $\mathscr{U}_{ad}$ the set of admissible controls defined by:

$$\mathscr{U}_{ad} = \Big\{ u : I \to U \text{ such that} u \text{ is measurable and } u(t) \in \Gamma(t),\ a.e. \Big\}.$$

Then, we have the following:

Theorem 3.1. [20] *$\mathscr{U}_{ad} \neq \emptyset$ and $\mathscr{U}_{ad} \subset L^2(I,U)$ is bounded, closed and convex. Also, $Cu \in L^2(I,U)$ for all $u \in \mathscr{U}_{ad}$.*

Definition 3.2. Let $u \in \mathscr{U}_{ad}$. A function $x \in \mathscr{C}(I;X)$ is called a mild solution of equation (1.1) if

$$x(t) = R(t)x_0 + \int_0^t R(t-s)\left[f(s,x(s)) + C(s)u(s)\right] ds \ \text{ for } t \in I \tag{3.1}$$

We have the following theorem on existence of mild solutions to equation (1.1) with respect to a given control $u \in \mathscr{U}_{ad}$.

Theorem 3.3. *Assume that $(\mathbf{H_1}) - (\mathbf{H_5})$ hold. Then for each $u \in \mathscr{U}_{ad}$, equation (1.1) has a unique mild solution on $[0,b]$.*

Proof. Let $b_1,\ \rho > 0$, and $x \in X$ such that $\|x\| \leq \rho$. For $t \in [0,b_1]$, we have by the local Lipschitz condition on f that

$$\|f(t,x)\| \leq L_f(\rho)\|x\| + \|f(t,0)\| \leq L_f(\rho)\rho + \sup_{s \in [0,b_1]} \|f(s,0)\|.$$

b_1 will be chosen sufficiently small enough to get the local existence of mild solutions.

Let $x \in X,\ \rho = \|x\| + 1$ and $\rho^* = L_f(\rho)\rho + \sup_{s\in[0,b_1]} \|f(s,0)\|$.

We define the following space

$$E_0 = \Big\{ x \in \mathscr{C}([0,b_1];X) \text{ such that } \sup_{s\in[0,b_1]} \|x(s) - x(0)\| \le 1 \Big\}.$$

Then, E_0 is a closed subset of $\mathscr{C}([0,b_1];X)$ which is endowed with the uniform norm topology. Let

$$M_b = \sup_{t\in[0,b]} \|R(t)\|.$$

Define the operator $K : E_0 \to \mathscr{C}([0,b_1];X)$ by

$$(Kx)(t) = R(t)x_0 + \int_0^t R(t-s)\big[f(s,x(s)) + C(s)u(s)\big]\, ds \ \text{ for } t \in [0,b_1]$$

We claim that $K\Big(E_0\Big) \subset E_0$. In fact let $x \in E_0$ and $t \in [0,b_1]$. Then,

$$\begin{aligned} \|(Kx)(t) - x(0)\| &\le \|R(t)x(0) - x(0)\| \\ &\quad + \int_0^t \Big\| R(t-s)\big[f(s,x(s)) + C(s)u(s)\big] \Big\| ds \\ &\le \|R(t)x(0) - x(0)\| + M_b\rho^* t + M_b \|C\| \, \|u\|_{L^2}\sqrt{t}. \end{aligned}$$

Now, choose b_1 sufficiently small such that

$$\sup_{s\in[0,b_1]} \Big\{ \|R(s)x(0) - x(0)\| + M_b\rho^* s + M_b \|C\| \, \|u\|_{L^2}\sqrt{s} \Big\} < 1. \tag{3.2}$$

Consequently,

$$\|(Kx)(t) - x(0)\| \le \|R(t)x(0) - x(0)\| + M_b\rho^* t + M_b \|C\| \, \|u\|_{L^2}\sqrt{t} < 1 \ \text{ for } t \in [0,b_1].$$

Hence, $K\Big(E_0\Big) \subset E_0$.

Let $x, y \in E_0$ and $t \in [0,b_1]$. Then, there exists $\rho > 0$ such that $\|x\|,\ \|y\| \le \rho$. We have that

$$\begin{aligned} \|(Kx)(t) - (Ky)(t)\| &\le M_b \int_0^t \|f(s,x(s)) - f(s,y(s))\| \, ds \\ &\le M_b L_f(\rho) \int_0^t \|x(s) - y(s)\| \, ds \\ &\le M_b L_f(\rho) \int_0^t \sup_{\tau\in[0,s]} \|x(\tau) - y(\tau)\| \, d\tau \\ &\le M_b L_f(\rho)\, b_1 \, \|x - y\| \end{aligned}$$

Now, since

$$M_b L_f(\rho)\, b_1 \le M_b \rho^* b_1 < \sup_{s\in[0,b_1]} \left\{ \|R(s)x(0)-x(0)\| + M_b \rho^* s + M_b \|C\| \|u\|_{L^2} \sqrt{s} \right\}.$$

Condition (3.2) implies that

$$M_b L_f(\rho)\, b_1 < 1.$$

Thus, K is a strict contraction on E_0. It follows from the contraction mapping principle that K has a unique fixed point $x \in E_0$, which is the unique mild solution of equation (1.1) with respect to u on $[0,b_1]$.

Using the same arguments, we can show that x can be extended to a maximal interval of existence $[0,t_{\max}[$.

Lemma 3.4. [4] *If* $t_{\max} < b$*, then,* $\limsup_{t\to t_{\max}} \|x(t)\| = \infty$.

We show that $t_{\max} = b$.

Assume on the contrary that $t_{\max} < b$. Then for $t \in [0,t_{\max}]$ we have that

$$x(t) = R(t)x_0 + \int_0^t R(t-s)\left[f(s,x(s)) + C(s)u(s)\right] ds.$$

It follows that

$$\begin{aligned}
\|x(t)\| &\le M_b\|x_0\| + M_b \int_0^t \|f(s,x(s))\|\, ds + M_b \int_0^t \|C(s)u(s)\|\, ds \\
&\le M_b\|x_0\| + M_b t_{\max} a_f + M_b a_f \int_0^t \|x(s)\|\, ds + M_b\|C\| \int_0^t \|u(s)\|\, ds \\
&\le M_b\|x_0\| + M_b t_{\max} a_f + M_b \sqrt{t_{\max}}\, \|C\| \|u\|_{L^2} + M_b a_f \int_0^t \|x(s)\|\, ds
\end{aligned}$$

This implies that

$$\|x(t)\| \le M_b\|x_0\| + M_b t_{\max} a_f + M_b \sqrt{t_{\max}}\, \|C\| \|u\|_{L^2} + M_b a_f \int_0^t \|x(s)\|\, ds$$

It follows by Gronwall's inequality that

$$\|x(t)\| \le \beta^* e^{M_b a_f t} \quad \text{for } t \in [0,t_{\max}],$$

where $\beta^* = M_b\|x_0\| + M_b t_{\max} a_f + M_b \sqrt{t_{\max}}\, \|C\| \|u\|_{L^2}$.

Thus

$$\lim_{t\to t_{\max}} \|x(t)\| \le \beta^* e^{M_b a_f t_{\max}} < \infty.$$

This contradicts Lemma 3.4. Therefore, $t_{\max} = b$ and hence, equation (1.1) has a unique mild solution on $[0,b]$.

□

4. Continuous Dependence and Existence of the Optimal Control Solving Equation (1.1)

In this section, we discuss the continuous dependence of the mild solutions of equation (1.1) on the controls and initial states, and the existence of solutions of the Lagrange problem associated to equation (1.1).
We have the following a priori estimation.

Lemma 4.1. *Suppose* $(\mathbf{H_1})-(\mathbf{H_3})$ *holds and assume that equation (1.1) has a mild solution* x_u *on* $[0,b]$ *with respect to* $u\in\mathscr{U}_{ad}$*. Then, there exists a constant* $\rho>0$ *independent of* u *such that* $\|x_u(t)\|\le\rho \quad$ *for* $t\in[0,b]$*, (*ρ *depends only on* $\mathscr{U}_{ad}$ *and* x_0*).*

Proof. Let

$$t\in[0,b],\; M_b=\sup_{t\in[0,b]}\|R(t)\| \text{ and } \|C\|=\sup_{t\in I}\|C(t)\|_{\mathscr{L}(U,X)}.$$

Since $\mathscr{U}_{ad}$ is bounded, let $\widetilde{K}>0$ be such that $\|u\|_{L^2}\le\widetilde{K}$ for all $u\in\mathscr{U}_{ad}$. Then, we have that

$$\begin{aligned}\|x(t)\| &\le M_b\|x_0\|+M_b\int_0^t\|f(s,x(s))\|\,ds+M_b\int_0^t\|C(s)u(s)\|\,ds\\ &\le M_b\|x_0\|+M_b\,b\,a_f+M_b a_f\int_0^t\|x(s)\|\,ds+M_b\|C\|\int_0^b\|u(s)\|\,ds\\ &\le M_b\|x_0\|+M_b\,b\,a_f+M_b\sqrt{b}\,\|C\|\|u\|_{L^2}+M_b\,a_f\int_0^t\|x(s)\|\,ds\\ &\le M_b\|x_0\|+M_b\,b\,a_f+M_b\sqrt{b}\,\|C\|\widetilde{K}+M_b\,a_f\int_0^t\|x(s)\|\,ds.\end{aligned}$$

Thus

$$\|x(t)\|\le M_b\|x_0\|+M_b\,b\,a_f+M_b\sqrt{b}\,\|C\|\widetilde{K}+M_b\,a_f\int_0^t\|x(s)\|\,ds. \tag{4.1}$$

It follows by Gronwall's inequality that

$$\|x(t)\|\le Me^{b\,a_f M_b}=:\widetilde{M},$$

with $M=M_b\|x_0\|+M_b\,b\,a_f+M_b\sqrt{b}\,\|C\|\widetilde{K}$.
As a result, for $t\in I$, we have

$$\|x_u(t)\|\le\widetilde{M}:=\rho$$

That is $\|x_u(t)\|\le\rho$ for all $t\in I$. This completes the proof of the Lemma.

□

We have the following theorem on continuous dependence of the mild solutions of equation (1.1) on the controls and initial states.

Theorem 4.2. *For all $\lambda > 0$, there exists $\gamma^*(\lambda) > 0$ such that for all $x_0^1,\ x_0^2 \in B(0,\lambda)$,*

$$\|x^1(t) - x^2(t)\| \leq \gamma^* \left(\|x_0^1 - x_0^2\| + \|u^1 - u^2\|_{L^2} \right) \ \textit{for} \ \ t \in [0,b],$$

where

$$x^i(t) = R(t)x_0^i + \int_0^t R(t-s) \left[f(s, x^i(s)) + C(s)u^i(s) \right] ds \ \ \textit{for}\ t \in I \tag{4.2}$$

and $u^i \in \mathscr{U}_{ad}$, for $i = 1,2$.

Proof. Let x^i, for $i = 1,2$, be two mild solutions of equation (1.1), corresponding to the controls $u^i \in \mathscr{U}_{ad}$ and $\lambda > 0$ such that $x_0^1,\ x_0^2 \in B(0,\lambda)$.

$$x^i(t) = R(t)x_0^i + \int_0^t R(t-s) \left[f(s, x^i(s)) + C(s)u^i(s) \right] ds \ \ \text{for}\ t \in I$$

By Lemma 4.1, we have that, there exists a constant $\rho_\lambda = \widetilde{M} > 0$ such that $\|x^i(s)\| \leq \rho_\lambda,\ i = 1, 2.$
Now, for $t \in [0,b]$, we have

$$\begin{aligned}
\|x^1(t) - x^2(t)\| &\leq M_b \|x_0^1 - x_0^2\| + M_b \int_0^t \|f(s, x^1(s)) - f(s, x^2(s))\| \, ds \\
&+ M_b \int_0^t \|C(s)u^1(s) - C(s)u^2(s)\| \, ds \\
&\leq M_b \|x_0^1 - x_0^2\| + M_b L_f(\rho_\lambda) \int_0^t \left(\|x^1(s) - x^2(s)\| \right) ds \\
&+ M_b \int_0^t \|C(s)u^1(s) - C(s)u^2(s)\| \, ds \\
&\leq M_b \|x_0^1 - x_0^2\| + M_b L_f(\rho_\lambda) \int_0^t \|x^1(s) - x^2(s)\| \, ds \\
&+ M_b L_f(\rho_\lambda) \sqrt{b} \|C\| \, \|u^1 - u^2\|_{L^2}
\end{aligned}$$

That is

$$\|x^1(t) - x^2(t)\| \leq M_b \|x_0^1 - x_0^2\| + M_b L_f(\rho_\lambda) \sqrt{b} \|C\| \, \|u^1 - u^2\|_{L^2} + M_b L_f(\rho_\lambda) \int_0^t \|x^1(s) - x^2(s)\| \, ds. \tag{4.3}$$

By Gronwall's inequality, we have that

$$\|x^1(t)-x^2(t)\| \quad \leq \quad \Big[M_b\|x_0^1-x_0^2\|+M_bL_f(\rho_\lambda)\sqrt{b}\,\|C\|\,\|u^1-u^2\|_{L^2}\Big]e^{M_bL_f(\rho_\lambda)\,b}.$$

This implies that

$$\|x^1(t)-x^2(t)\| \quad \leq \quad \Big[M_b\|x_0^1-x_0^2\|+M_bL_f(\rho_\lambda)\sqrt{b}\,\|C\|\,\|u^1-u^2\|_{L^2}\Big]e^{M_bL_f(\rho_\lambda)\,b},$$

Let

$$\gamma^*(\lambda) := \max\Big\{M_b\,e^{M_bL_f(\rho_\lambda)b},\ M_bL_f(\rho_\lambda)\sqrt{b}\,\|C\|\,e^{M_bL_f(\rho_\lambda)\,b}\Big\}.$$

Then, we have that

$$\|x^1(t)-x^2(t)\| \leq \gamma^*(\lambda)\Big(\|x_0^1-x_0^2\|+\|u^1-u^2\|_{L^2}\Big)\ \text{ for } t\in[0,b].$$

And the proof is complete.

□

Now, we study the existence of solutions to the following Lagrange problem

$$(\mathscr{L}\mathscr{P})\begin{cases}\text{Find a control } u^0\in\mathscr{U}_{ad}\ \text{ such that}\\[2em] \mathscr{J}(u^0)\leq\mathscr{J}(u)\ \text{ for all }\ u\in\mathscr{U}_{ad},\end{cases}$$

where

$$\mathscr{J}(u) := \int_0^b \mathscr{L}\Big(t, x^u(t), u(t)\Big)\,dt,$$

and x^u denotes the mild solution of (1.1) corresponding to the control $u\in\mathscr{U}_{ad}$ and the initial data x_0.

For the existence of solutions to problem $(\mathscr{L}\mathscr{P})$, we make the following assumptions.

$(\mathbf{H_L})$

- **(i)** The functional $\mathscr{L} : I\times X\times U\to\mathbb{R}\cup\{\infty\}$ is Borel measurable.
- **(ii)** $\mathscr{L}(t,\cdot,\cdot)$ is sequencially lower semicontinuous on $X\times U$ for almost all $t\in I$.
- **(iii)** $\mathscr{L}(t,y,\cdot)$ is convex on U for each $y\in X$ and almost all $t\in I$.
- **(iv)** There exist constants $\beta\geq 0,\ \gamma>0$, and $\mu\in L^1(I)$ nonnegative such that

$$\mathscr{L}(t,y,u)\geq\mu(t)+\beta\|y\|+\gamma\|u\|.$$

We have the following result on the existence of optimal controls for problem $(\mathscr{L}\mathscr{P})$.

Theorem 4.3. *Assume that hypotheses* $(\mathbf{H_1})-(\mathbf{H_5})$ *and* $(\mathbf{H_L})$ *hold. Then the Lagrange problem* $(\mathscr{L}\mathscr{P})$ *admits at least one optimal pair, that is there exists an admissible control pair* $(x^0,u^0)\in\mathscr{C}([0,b],X)\times\mathscr{U}_{ad}$ *such that*

$$\mathscr{J}(u^0)=\int_0^b\mathscr{L}\Big(t,x^0(t),u^0(t)\Big)\,dt\leq\int_0^b\mathscr{L}\Big(t,x^u(t),u(t)\Big)\,dt=\mathscr{J}(u)\ \text{ for } u\in\mathscr{U}_{ad}.$$

Proof. If $\inf\Big\{\mathscr{J}(u) : u \in \mathscr{U}_{ad}\Big\} = \infty$, we are done.
Without loss of generality, assume that $\inf\Big\{\mathscr{J}(u) : u \in \mathscr{U}_{ad}\Big\} = \delta < \infty$.
Suppose that $\delta = -\infty$, then for each $n \in \mathbb{N}$, there exists $(u^n)_{n\geq 1} \subset \mathscr{U}_{ad}$ such that

$$\mathscr{J}(u^n) < -n \tag{$*$}$$

Boundedness of $\mathscr{U}_{ad}$ implies that $(u^n)_{n\geq 1}$ is bounded and so there exists a subsequence $(u^{n_k})_{k\geq 1}$ of $(u^n)_{n\geq 1}$ that converges weakly to some u^0 in $L^2(I,U)$, since $L^2(I,U)$ is reflexive. But $\mathscr{U}_{ad}$ is closed and convex, so by Mazur's Theorem, it is weakly closed and therefore, $u^0 \in \mathscr{U}_{ad}$. By hypothesis $(\mathbf{H_L})$, $\mathscr{L}(t,y,\cdot)$ is weakly lower semicontinuous, so we have that

$$\mathscr{L}(t,y,u^0) \leq \liminf_{k\to\infty} \mathscr{L}(t,y,u^{n_k}) < -\infty,$$

which implies that $\mathscr{J}(u^0) < -\infty$ using $(*)$. And this is a contradiction since $\mathscr{J}(u^0) \in \mathbb{R}\cup\{\infty\}$. Hence $\delta \in \mathbb{R}$.
Now by the definition of δ, there exists a minimizing sequence, a feasible pair $((x^n,u^n))_{n\geq 1} \subset \mathscr{S}_{ad}$ such that

$$\int_0^b \mathscr{L}\Big(t, x^n(t), u^n(t)\Big)\, dt \longrightarrow \delta \text{ as } n \to \infty,$$

where

$$\mathscr{S}_{ad} := \Big\{(x,u) : x \text{ is a mild solution of equation (1.1) corresponding to the control } u \in \mathscr{U}_{ad}\Big\}.$$

Boundedness of $\mathscr{U}_{ad}$ and the fact that $L^2(I,U)$ is reflexive imply that $(u^n)_{n\geq 1}$ has a subsequence denoted for simplicity by $(u^k)_{k\geq 1}$, that converges weakly to some u^0 in $L^2(I,U)$. But $\mathscr{U}_{ad}$ is closed and convex, so by Mazur's Theorem, it is weakly closed and therefore, $u^0 \in \mathscr{U}_{ad}$.
Let

$$x^k(t) = R(t)x_0 + \int_0^t R(t-s)\Big[f(s,x^k(s)) + C(s)u^k(s)\Big]\, ds \text{ for } t \in I$$

denote the subsequence of $(x^n)_{n\geq 1}$ corresponding to the control sequence $(u^k)_{k\geq 1}$ and x^0 be the mild solution corresponding to the control $u^0 \in \mathscr{U}_{ad}$. We show that $x^k \to x^0$.

For $t \in [0,b]$, we have

$$\begin{aligned}
\left\|x^k(t)-x^0(t)\right\| &\leq \int_0^t \left\|R(t-s)\left[f(s,x^k(s))-f(s,x^0(s))\right]\right\| ds \\
&+ \int_0^t \left\|R(t-s)\left[C(s)u^k(s)-C(s)u^0(s)\right]\right\| ds \\
&\leq M_b L_f(\rho)\int_0^t \left\|x^k(s)-x^0(s)\right\| ds + M_b\int_0^t \left\|C(s)u^k(s)-C(s)u^0(s)\right\| ds \\
&\leq M_b L_f(\rho)\int_0^t \left\|x^k(s)-x^0(s)\right\| ds + M_b\sqrt{b}\left(\int_0^t \left\|C(s)u^k(s)-C(s)u^0(s)\right\|^2 ds\right)^{\frac{1}{2}} \\
&\leq M_b L_f(\rho)\int_0^t \left\|x^k(s)-x^0(s)\right\| ds + M_b\sqrt{b}\left\|Cu^k-Cu^0\right\|_{L^2(I,U)}
\end{aligned}$$

That is

$$\left\|x^k(t)-x^0(t)\right\| \leq M_b L_f(\rho)\int_0^t \left\|x^k(s)-x^0(s)\right\| ds + M_b\sqrt{b}\left\|Cu^k-Cu^0\right\|_{L^2(I,U)} \tag{4.4}$$

By Gronwall's inequality, we have that

$$\left\|x^k(t)-x^0(t)\right\| \leq M^{**}\left\|Cu^k-Cu^0\right\|_{L^2(I,U)}, \text{ where } M^{**} = M_b\sqrt{b}\,e^{M_b b L_f(\rho)}. \tag{4.5}$$

We have the following Lemma.

Lemma 4.4. [20] *Let $(u^n)_{n\geq 1} \subset \mathscr{U}_{ad}$ and $u^0 \in \mathscr{U}_{ad}$ such that $(u^n)_{n\geq 1}$ converges weakly to u^0. Then,*

$$\left\|Cu^k-Cu^0\right\|_{L^2(I,U)} \longrightarrow 0 \text{ as } k\to\infty, \text{ if } C \in L^\infty(I;\mathscr{L}(U,X)).$$

We have by (4.5) that

$$\left\|x^k-x^0\right\| \leq M^{**}\left\|Cu^k-Cu^0\right\|_{L^2(I,U)},$$

and therefore, it follows by Lemma 4.4 that

$$x^k \longrightarrow x^0 \text{ as } k\to\infty.$$

We note that $(\mathbf{H_L})$ implies the assumptions of Balder's Theorem. Hence by using Balder's Theorem, we can conclude that $(x,u) \mapsto \int_0^b \mathscr{L}\Big(t, x(t), u(t)\Big)\, dt$ is sequencially lower semicontinuous in the strong topology of $L^1(I,X)\times L^1(I,U)$.

Now, since $L^2(I,X)\times L^2(I,U) \subset L^1(I,X)\times L^1(I,U)$, $\mathscr{J}$ is also sequencially lower semicontinuous on $L^2(I,X)\times L^2(I,U)$, and in the strong topology of $L^1(I,X\times U)$.

Hence, $\mathscr{J}$ is weakly lower semicontinuous on $L^2(I,U)$, and since by $(\mathbf{H_L})-(\mathbf{iv})$, $\mathscr{J} > -\infty$, $\mathscr{J}$ attains its infimum at $u^0 \in \mathscr{U}_{ad}$, that is

$$\delta = \lim_{k\to\infty}\int_0^b \mathscr{L}\Big(t, x^k(t), u^k(t)\Big)\, dt \geq \int_0^b \mathscr{L}\Big(t, x^0(t), u^0(t)\Big)\, dt = \mathscr{J}(u^0) \geq \delta.$$

Thus, $\delta = \mathscr{J}(u^0)$, and hence there exists an admissible control $u^0 \in \mathscr{U}_{ad}$ such that

$$\mathscr{J}(u^0) \leq \mathscr{J}(u) \ \text{ for all } \ u \in \mathscr{U}_{ad}.$$

This completes the proof.

□

we now illustrate our main result by the following example.

5. Example

Let Ω be bounded domain in $\mathbb{R}^n$ with smooth boundary and consider the following nonlinear integrodifferential equation.

(5.1)

$$\begin{cases} \frac{\partial v(t,\xi)}{\partial t} = \Delta v(t,\xi) + \int_0^t \zeta(t-s)\Delta v(s,\xi)\, ds + \alpha(t)\sin(v^2(t,\xi)) + \beta(t)\omega(t,\xi) \text{ for } t\in[0,1]=I \text{ and } \xi\in\Omega \\ v(t,\xi) = 0 \text{ for } t\in[0,1] \text{ and } \xi\in\partial\Omega \\ v(0,\xi) = v_0(\xi), \end{cases}$$

where $\alpha,\ \beta \in \mathscr{C}(I,\mathbb{R})$, $\omega : I\times\Omega \to \mathbb{R}$ continuous in t, and $\zeta \in W^{1,1}(\mathbb{R}^+,\mathbb{R})$.

Let $X = U = L^2(\Omega)$. For $\eta > 0$, we define the set of admissible controls $\mathscr{U}_{ad}$ by

$$\mathscr{U}_{ad} := \Big\{u : I \to U \text{ such that } u \text{ is measurable and } \|u\|_{L^2(I,U)} \leq \eta\Big\},$$

where

$$\|u\|^2_{L^2(I,U)} = \int_0^1 \left(\int_\Omega u^2(s)(\xi)\, d\xi\right) ds.$$

We define $A : \mathscr{D}(A) \subset X \to X$ by:

$$\begin{cases} \mathscr{D}(A) = H^2(\Omega) \cap H_0^1(\Omega) \\ \\ Av = \Delta v \ \text{ for } v \in \mathscr{D}(A). \end{cases}$$

Theorem 5.1. *(***Theorem 4.1.2, p. 79 of** [19]*) A is the infinitesimal generator of a C_0-semigroup on $L^2(\Omega)$.*

A generates a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on $L^2(\Omega)$.

Define

$$x(t)(\xi) \ = \ v(t,\xi), \quad x'(t)(\xi) \ = \ \frac{\partial v(t,\xi)}{\partial t}, \quad \omega(t,\xi) = u(t)(\xi).$$

$$f(t,x(t))(\xi) \ = \ \alpha(t)\sin\big(x^2(t)(\xi)\big) \ \text{ for } t \in [0,1] \text{ and } \xi \in \Omega.$$

$C(t): X \to X$ be defined by $\Big(C(t)u(t)\Big)(\xi) = C(t)u(t)(\xi) = \beta(t)\omega(t,\xi)$.

$$(B(t)x)(\xi) \ = \ \zeta(t)\Delta v(t,\xi) \ \text{ for } \ t \in [0,1], \ \ x \in \mathscr{D}(A) \text{ and } \ \xi \in \Omega.$$

Equation (5.1) is then transformed into the following form

$$\begin{cases} x'(t) = Ax(t) \ + \ \displaystyle\int_0^t B(t-s)x(s)ds \ + \ f(t,x(t)) + C(t)u(t) \ \text{ for } t \in I = [0,1], \\ \\ x(0) = x_0. \end{cases} \tag{5.2}$$

One can see that, f satisfies $(\mathbf{H_3})$. Now we consider the following cost function:

$$\mathscr{J}(u) := \int_0^1 \mathscr{L}\Big(t, x^u(t), u(t)\Big)\, dt,$$

where $\mathscr{L} \,:\, [0,1] \times L^2(\Omega) \times L^2(\Omega) \longrightarrow \mathbb{R}$ is defined by:

$$\mathscr{L}\Big(t, x, u\Big) \quad = \quad \|x\| + \|u\|.$$

$\mathscr{L}$ satisfies all the conditions of hypothesis $(\mathbf{H_L})$. Then,

$$\mathscr{J}(u) = \int_0^1 \Big(\|x^u(t)\| + \|u(t)\|\Big)\, dt.$$

Hence, all the conditions of Theorem 4.3 are satisfied, and therefore, equation (5.2) has at least one optimal pair.

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