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FIXED POINT THEOREMS IN MENGER SPACE USING THE NOTION OF COMPATIBILITY AND SUBSEQUENTIALLY CONTINUITY

RAMESH BHINDE[1,*], V.K. GUPTA[2], ARIHANT JAIN[3]

[1]Govt.P.G.College, Alirajpur (M.P.), India

[2]Dept. of Mathematics, Govt. Madhav Science P.G. College, Ujjain (M.P.), India

[3]Dept. of Applied Mathematics, Shri Guru Sandipani Institute of Technology and Science, Ujjain (M.P.), India

Abstract. In this paper is to prove a common fixed point theorem for four mappings using the notion of compatibility and sub sequentially continuity in Menger space.

Keywords: fix point; Menger space; compatibility; subsequently continuity.

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1. INTRODUCTION

In 1942, Professor Karl Menger [11] has introduced the theory of probabilistic metric space in which a distribution function was used instead of non-negative real number as value of the metric. The notion of PM-space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. In 1960, Schweizer and Sklar [14] studied this concept and gave fundamental result on this space. Fixed point theory is one of the fruitful and effective tools in mathematics.

In 1986, Jungck [8] introduced the notion of Compatible maps for a pair of self-maps in metric space. In 1991, Pant [13] noticed these criteria for fixed points of contraction mappings and introduced a new continuity condition, known as reciprocal continuity and obtained a common fixed point theorem by using the compatibility in metric spaces. Healso showed that in the setting of common fixed point theorems for compatible mappings satisfying contraction

[*]Corresponding author

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conditions, the notion of reciprocal continuity is weaker than the continuity of one of the mappings.

In 1998, Jungck and Rhoades [9] introduced the concept of weakly compatibility and showed that each pair of compatible maps is weakly compatible but the converse need not to be true. In 2005 Singh and Jain [15] generalized the result of Mishra [12] using the concept of weak compatibility and compatibility of pair of self-maps.

In 2008 Al-Thagafi and shahzad [1] introduced the concept of occasionally weakly compatible (OWC) mappings in metric space which is the most general concept among all the commutativity concepts. In 2012, Doric et.al [6] shown that the condition of occasionally weak compatibility reduced to weak compatibility. Bouhadjera and Godet-Thobie [2] introduced two new notion namely subsequential continuity and subcompatibility which are weaker than reciprocal continuity and compatibility respectively. Further Imdad et al. [7]improved the result of Bouhadjera and Godet-Thobie [2].

The object of this paper is to prove a common fixed point theorem using the notion of compatibility and sub sequentially continuity in Menger space.

2. PRELIMINARY NOTES

Definition 2.1(Schweizer and Sklar [14]) A Mapping $F:R\rightarrow R^+$ is said to be a distribution function if it is non-decreasing and leftcontinuous with

$\text{Inf}\,\{F(t):t\in R\} = 0$ and $\text{Sup}\,\{F(t):t\in R\} = 1$

We will denote the Δ the set of all distribution function defined on $[-\infty,\infty]$ while H(t) will always denote thespecific distribution function defined by

$$H(t)= \begin{cases} 0\,, \text{if } t \leq 0 \\ 1, \ \text{if } t > 0 \end{cases}$$

If X is a non-empty set, $F : X\times X \rightarrow \Delta$ is called a probabilistic distance on X and the value of F at $(x, y) \in X \times X$ is represented by $F_{x,y}$.

Definition 2.2(Schweizer and Sklar [14]) The ordered pair (X,F) is called a probabilistic metric space (shortly PM-space) if X is nonempty set and F is a probabilistic distance satisfying the following conditions:

for all $x,y,z\in X$ and$t,s> 0$

PM-1 $F_{x,y}(t) = 1$ if and only if $x=y$

PM-2 $F_{x,y}(0) = 0$

PM-3 $F_{x,y}(t) = F_{y,x}(t)$

PM-4 If $F_{x,z}(t) = 1$ and $F_{z,y}(s) = 1$ then $F_{x,y}(t+s)=1$

the ordered triple (X,F,Δ) is called Menger space if (X,F) is PM space and Δ is a triangular norm such that for all x,y,z∈X and t,s>0

PM-5 $F_{x,y}(t+s) \geq F_{x,z}(t) + F_{z,y}(s)$

Definition 2.3(Schweizer and Sklar [14]) A Menger space (X, F, Δ) with the continuous t-norm Tis said to be complete iff every Cauchy sequence in X converges to a point in X.

Definition 2.4(Mishra [12]) Twoself maps A and S of a Menger Space (X,F,Δ) are said to be compatible if

$F_{ASxn,SAxn}(t) \to 1$ for all t>0 Whenever {xn} is a sequence in X such that Axn,Sxn→ z for some z ∈X as n → ∞ .

Definition 2.5(Singh and Jain [16]) Two self-maps A and S of a non-empty set X are saidto be weakly compatible (or coincidentally commuting) if they commute at theircoincidence points, i.e. if Az= Sz for some z∈ X, then ASz= SAz.

Definition.2.6(Jungck[10]) Two self-mappings A and S of non-empty set X are occasionally weakly compatible(OWC) if and only if there exist a point z∈ X which is coincidence point of A and S at which A and S commute.

Definition.2.7(Bouhadjera and Godet-Thobie [2])A pari of self-mappings(A,S) is said to be sub compatible on a Menger space(X,F,Δ)iff there exist a sequence $\{x_n\}$in X such that

$$\lim_{n\to\infty} Axn = \lim_{n\to\infty} Sxn = x \qquad \text{for some } x \in X$$

$$\text{and} \quad \lim_{n\to\infty} F_{ASx_n,SAx_n}(t) = 1 \quad \text{for all } t > 0$$

Definition.2.8A pair of self-mappings(A,S) is said to be subsequentially continuous on a Menger space(X,F,Δ) if and only if there exist a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Axn = \lim_{n\to\infty} Sxn = x \qquad \text{for some } x \in X$$

$$\text{and} \quad \lim_{n\to\infty} ASxn = Ax \text{ and } \lim_{n\to\infty} SAxn = Sx$$

Lemma 2.9 Let (X,F,Δ)be a Menger space. If there exists k ∈(0, 1) such that

$F_{x,y}(kt) \geq F_{x,y}(t)$,for all x, y ∈ X and t > 0

then x = y.

3. MAIN RESULT

Theorem 3.1 Let A, B, S and Tbe self maps on a Menger space (X, F, Δ) with continuous t-norm and if the pairs (A,S) and (B,T) are compatible and subsequentially continuous mappings then

(i) the pair (A,S) and (B,T) have a coincidence point,

(ii)there exist a constant k ∈(0, 1)such that

for all x, y ∈ X and t > 0

$$F_{Ax,By} \geq \min\{F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t), F_{Ax,Ty}(t), F_{By,Sx}(t)\}$$

Then A, B, S and T have a unique common fixed point in X.

Proof. Since the pair (A,S) and (B,T) is compatible and subsequentially continuous mappings, then from the definition there exist a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z \qquad \text{for some } z \in X$$

$$\text{and} \quad \lim_{n\to\infty} F_{ASx_n,SAx_n}(t) = F_{Az,Sz}(t) = 1 \quad \text{for all } t > 0$$

Then Az = Sz.Hence z is a coincidence point of pair (A,S).

Again, since (B,T) is compatible and subsequentially continuous mappings ,then from the definition, there exist a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = w \qquad \text{for some } w \in X$$

$$\text{and} \quad \lim_{n\to\infty} F_{BTy_n,TBy_n}(t) = F_{BTy_n,TBy_n}(t) = 1 \quad \text{for all } t > 0$$

then Bw= Tw. Hence w is a coincidence point of pair (B,T).

Step1. By taking $x = x_n$and y = y_n in (ii), we have

$$\mathbf{F_{Ax_n,By_n}(kt) \geq \min\{F_{Sx_n,Ty_n}(t), F_{Ax_n,Sx_n}(t), F_{By_n,Ty_n}(t), F_{Ax_n,Ty_n}(t), F_{By_n,Sx_n}(t)\}.}$$

Taking limit as n → ∞, we get

$$F_{z,w}(kt) \geq \min\{F_{z,w}(t), F_{z,z}(t), F_{w,w}(t), F_{z,w}(t), F_{w,z}(t)\}.$$

$$F_{z,w}(kt) \geq \min\{F_{z,w}(t), 1, 1, F_{z,w}(t), F_{w,z}(t)\}.$$

$$F_{z,w}(kt) \geq F_{z,w}(t)$$

From lemma2.9, we have z = w

Step2. By taking $x = z$and y = y_n in (ii), we have

$$\boldsymbol{F_{Az,By_n}(kt) \geq \min\{F_{Sz,Ty_n}(t), F_{Az,Sz}(t), F_{By_n,Ty_n}(t), F_{Az,Ty_n}(t), F_{By_n,Sz}(t)\}.}$$

Taking limit as n → ∞, we get

$F_{Az,w}(kt) \geq \min\{F_{Az,w}(t), F_{Az,Az}(t), F_{w,w}(t), F_{Az,w}(t), F_{w,Az}(t)\}.$

$F_{Az,w}(kt) \geq \min\{F_{Az,w}(t), 1, 1, F_{Az,w}(t), F_{w,Az}(t)\}.$

$F_{Az,w}(kt) \geq F_{Az,w}(t)$

From lemma2.9, we have Az = w

Step3. By taking $x = x_n$and y = z in (ii), we have

$\boldsymbol{F_{Ax_n,Bz}(kt) \geq \min\{F_{Sx_n,Tz}(t), F_{Ax_n,Sx_n}(t), F_{Bz,Tz}(t), F_{Ax_n,Tz}(t), F_{Bz,Sx_n}(t)\}.}$

Taking limit as n → ∞, we get

$F_{z,Bz}(kt) \geq \min\{F_{z,Bz}(t), F_{z,z}(t), F_{Bz,Bz}(t), F_{z,Bz}(t), F_{Bz,z}(t)\}.$

$F_{z,Bz}(kt) \geq \min\{F_{z,Bz}(t), 1, 1, F_{z,Bz}(t), F_{Bz,z}(t)\}.$

$F_{z,Bz}(kt) \geq F_{z,Bz}(t)$

From lemma2.9, we have z = Bz

Therefore Az=Sz=Bz=Tz=z. i.e z is a common fixed point theorem of A,B,S and T.

Step4. For uniqueness,let u (z ≠ u) is another common fixed point of A,B,S and T then Au=Su=Bu=Tu=u

By taking x= $z$and y = u in 3.1.2, we have

$F_{Az,Bu}(kt) \geq \min\{F_{Sz,Tu}(t), F_{Az,Sz}(t), F_{Bu,u}(t), F_{Az,Tu}(t), F_{Bu,Sz}(t)\}$

$F_{z,u}(kt) \geq \min\{F_{z,u}(t), F_{z,z}(t), F_{u,u}(t), F_{z,u}(t), F_{u,z}(t)\}$

$F_{z,u}(kt) \geq F_{z,u}(t)$

From lemma2.9, we have z = u which is contradiction of our hypothesis is z ≠ u. Hence z is unique common fixed point.

Corollary 3.2 Let A and S be self-maps on a Menger space (X, F, Δ) with continuous t-norm and if the pairs (A,S) and (B,T) are compatible and subsequentially continuous mappings then

(i) the pair (A,S) has a coincidence point,

(ii)there exist a constant k ∈(0, 1)such that

for all x, y ∈ X and t > 0

$F_{Ax,Ay} \geq \min\{F_{Sx,Sy}(t), F_{Ax,Sx}(t), F_{Ay,Sy}(t), F_{Ax,Sy}(t), F_{Ay,Sx}(t)\}$

Then A and S have a unique common fixed point in X.

Example 3.3 Let X=[0,∞) and d be the usual metric on X and for each t∈[0, 1] define

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|} & , \quad \text{if } t > 0 \\ 0 & , \quad \text{if } t = 0 \end{cases} \text{ for all } x,y \in X$$

Clearly (X, F, Δ) be a Menger space where t-normΔ is defined by Δ(a,b)=min{a,b} for all a,b∈ [0, 1].

We define self-maps A and S on X

$$A(X) = \begin{cases} \frac{x}{4}, & \text{if } x \in [0,1] \\ 5x-4, & \text{if } x \in (1,\infty) \end{cases} \quad S(X) = \begin{cases} \frac{x}{5}, & \text{if } x \in [0,1] \\ 4x-3, & \text{if } x \in (1,\infty) \end{cases}$$

Consider a sequence $\{x_n\} = \{\frac{1}{n}\}$ in X. Then

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} (\frac{1}{4n}) = 0 = \lim_{n\to\infty} (\frac{1}{5n}) = \lim_{n\to\infty} S(x_n)$$

$$\text{now}, \quad \lim_{n\to\infty} AS(x_n) = \lim_{n\to\infty} A(\frac{1}{5n}) = \lim_{n\to\infty} (\frac{1}{20n}) = 0 = A(0)$$

$$\lim_{n\to\infty} SA(x_n) = \lim_{n\to\infty} S(\frac{1}{4n}) = \lim_{n\to\infty} (\frac{1}{20n}) = 0 = S(0)$$

$$\text{and} \quad \lim_{n\to\infty} F_{ASx_n,SAx_n}(t) = 1 \quad \text{for all } t > 0$$

Consider another sequence $\{x_n\} = \{1 + \frac{1}{n}\}$ in X. Then

$$\lim_{n\to\infty} A(x_n) = \lim_{n\to\infty} (5 + \frac{5}{n} - 4) = 1 = \lim_{n\to\infty} (4 + \frac{4}{n} - 3) = \lim_{n\to\infty} S(x_n)$$

$$now, \quad \lim_{n\to\infty} AS(x_n) = \lim_{n\to\infty} A(1 + \frac{4}{n}) = \lim_{n\to\infty} (5 + \frac{20}{n} - 4) = 1 \neq A(1)$$

$$\lim_{n\to\infty} SA(x_n) = \lim_{n\to\infty} S(1 + \frac{5}{n}) = \lim_{n\to\infty} (4 + \frac{20}{n} - 3) = 1 \neq S(1)$$

$$\text{but} \quad \lim_{n\to\infty} F_{ASx_n,SAx_n}(t) = 1 \quad \text{for all } t > 0$$

Thus the pair (A,S) is compatible and subsequentially continuous.

Conflict of Interests

The author declares that there is no conflict of interests.

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