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SOME FIXED POINT THEOREMS FOR G-METRIC SPACES WITH EXPANSION MAPPINGS

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Abstract. In this paper we prove some fixed point results for mapping satisfying expansive conditions on complete G-metric spaces. Also we showed that if G-metric space (X,G) is symmetric, then existence and uniqueness of those results follows from some simple fixed point theorem based on Banach contraction principal in usual metric space (X,d_G),where (X,d_G) the metric induced by the G-metric space (X,G).

Keywords: Metric space; generalized metric space; D-metric space; 2-metric space; semi compatible mapping.

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1. Introduction

The study of unique common fixed points of mappings satisfying certain contractive conditions has been at the centre of rigorous research activity. During the sixties, the notion of 2 -metric space was introduced by Gahler [3] as a generalization of usual notion of metric space (X,d). But many other authors proved that there is no relation between these two functions. For instance, Ha et al. [4] showed that 2 -metric need not be continuous function on its variable, where as the ordinary metric is. These considerations led by Dhage [2] in 1992 to introduce a new class of generalized metric spaces called D-metric space as a generalization of ordinary metric spaces (X,d). However Z. Mustafa and B. Sims [5] have demonstrated that most of the claims concerning the fundamental topological structure of D-metric space are incorrect. Alternatively, they have introduced [7] more appropriate notion of generalized metric space which called G-metric space. They generalized the concept of metric, in which the real number is assigned to

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every triplet of an arbitrary set. Based on the notion of generalized metric spaces, Mustafa et al. [6-8] obtained some fixed point theorems for mappings satisfying different contractive conditions.

2. Preliminary Notes

Definition2.1[7]-Let X be nonempty set and let $G : X \times X \times X \to R^{+}$ be a function satisfying

(G1) $G(x, y, z) = 0$ if $x = y = z$

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) + ...$

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangular property)

Then the function G is called a generalized metric space or more specifically a G -Metric on X and the pair (X, G) is G -Metric space.

Proposition 2.2[7] - Let (X, G) is a G - metric space. Then for any x, y, z, a in X , it follows

- $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$
- $G(x, y, y) \leq 2G(y, x, x)$
- $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$
- $G(x, y, z) \leq \frac{2}{3}\{G(x, y, a) + G(x, a, z) + G(a, y, z)\}$
- $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$

Definition 2.3[7]- Let (X, G) be a G - metric space, and let $\{x_n\}$ be a sequence of points of X . A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim G(x, x_n, x_m) = 0$ and one says that the sequence $\{x_n\}$ is G -convergent to x. Thus, that is $x_n \to 0$ in G -Metric space (X, G), then for $\varepsilon > 0$ there exist $N \in$N Such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$ (we mean by N the set of natural number).

Definition 2.4[7] - Let (X,G) be a G - metric space. Then for a sequence $\{x_n\}$ in X and a point $x \in X$, the following are equivalent:

- $\{x_n\}$ is convergent to x
- $G(x_n, x_n, x) \to 0$ as $n \to \infty$
- $G(x_n, x, x) \to 0$ as $n \to \infty$
- $G(x_n, x_m, x) \to 0$ as $n, m \to \infty$

Definition 2.5[7] - Let (X,G)be a G- metric space. A sequence $\{x_n\}$is called G-Cauchy if given $\varepsilon > 0$, there is $N \in \mathrm{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $l, m, n \geq N$. That is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Definition 2.6[7]- In a G - metric space,(X,G),the following are equivalent.

(1) The sequence $\{x_n\}$ is G-Cauchy sequence.

(2) For every $\varepsilon > 0$, there exist $N \in \mathrm{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.

Definition 2.7[7]- Let$(X,G)$and(X', G')be two G-metric space, and let $f:(X,G) \to (X', G')$ be a function, then f is said to be $G-$continuous at a point $a \in X$ iff, given $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$. A function f is $G-$continuous at X iff it is $G-$continuous at all $a \in X$.

Definition2.8[7]- A G-metric space (X,G) is called symmetric G-metric space if $G(x, y, y) = G(y, x, x)$for all $x, y \in X$.

Proposition 2.9[7]- Every G-metric space (X,G)induces a metric space (X, d_G) defined by

$$d_G(x, y) = G(x, y, y) + G(y, x, x),$$

For all $x, y \in X$.

Note that if$(X,G)$is symmetric, then

$$d_G(x, y) = 2G(x, y, y) \text{ for all } x, y \in X .$$

However, if (X,G) is not symmetric then it holds by the G metric properties that $\frac{3}{2}G(x,y,y) \le d_G(x,y) \le 3G(x,y,y)$ for $x,y \in X$. Also $G(x,y,y) \ge \frac{1}{3}d_G(x,y)$ for all $x,y \in X$.

Definition 2.10[7]- A G-metric space (X,G) is said to be $G-$complete if every G- Cauchy sequence in (X,G) is $G-$convergent in (X,G).

Proposition 2.11[7]- A G-metric space (X,G) is $G-$complete iff (X,d_G) is a complete metric space.

Lemma 2.12[7]- Let (X,G) is a G- metric space. If sequence $\{x_n\}$ in X converges to x and $\{y_n\}$ converges to y , then $\lim G(x_n,y_n,y_n) = G(x,y,y)$.

Lemma 2.13[7]- Let (X,G) is a G - metric space and $\{y_n\}$ is a any sequence satisfying $G(y_{n+1},y_{n+1},y_n) \le k^n G(y_0,y_1,y_1)$ where $k<1$, then $\{y_n\}$ is Cauchy sequence.

Definition 2.14- Let (X,d) be a metric space and $f\ \&\ g$ are self maps of X .if $\lim fx_n = \lim gx_n = t$ for some $t \in X$ then (f,g) is called semi compatible if $\lim fgx_n = gt$ holds.

Now we define semi compatibility on G metric space.

Definition 2.15- Let (X,G) be a G-metric space and f and g be two self maps of X . Then pair (f,g) is called semi compatible if whenever $\{x_n\}$ in X such that $\{fx_n\}$ and $\{gx_n\}$ are G - convergent to some $t \in X$ then $\lim G(fgx_n, gt, gt) = 0$.

Theorem 2.16- Let (X,d) be a complete metric space and $f\ \&\ g$ be a function mapping X into itself, satisfying the following conditions

(a) $\quad d(fx,fy) \ge ad(fx,gx) + bd(fy,gy) + cd(gx,gy)$

Where a,b,c are numbers, satisfy $a>1$, $c>1$, $b \in R$ $(a+b+c>1)$.

(b) Pair (f,g) is semi compatible with g is continuous.

Then, $f\ \&\ g$ have a unique common fixed point in X .

Proof- Let x_0 be any point in X .Then there exist point $x_1 \in X$ such that $fx_1 = gx_0$.We define a sequence $fx_{n+1} = gx_n = y_n$ where $n = 0,1,2...$. Now by using (a)

$d\left(fx_n, fx_{n+1}\right) \geq ad\left(fx_n, gx_n\right) + bd\left(fx_{n+1}, gx_{n+1}\right) + cd\left(gx_n, gx_{n+1}\right)$
$d\left(y_{n-1}, y_n\right) \geq ad\left(y_{n-1}, y_n\right) + bd\left(y_n, y_{n+1}\right) + cd\left(y_n, y_{n+1}\right)$

$d\left(y_{n+1}, y_n\right) \leq \frac{1-a}{b+c} d\left(y_{n-1}, y_n\right)$. Since $\frac{1-a}{b+c} < 1$ therefore $a+b+c > 1$.

Let $p = \frac{1-a}{b+c}$ then $d\left(y_{n+1}, y_n\right) \leq pd\left(y_{n-1}, y_n\right)$, by similar argument it yields

$d\left(y_{n+1}, y_n\right) \leq p^n d\left(y_0, y_1\right)$ ...(1)

Now we prove that $\{y_n\}$ is a Cauchy sequence. For some $m, n (m > n)$ we have

$d\left(y_n, y_m\right) \leq d\left(y_n, y_{n+1}\right) + d\left(y_{n+1}, y_{n+2}\right) + ... + d\left(y_{m-1}, y_m\right)$. Since $m > n$,let $m = n + k$ we have $d\left(y_n, y_m\right) \leq d\left(y_n, y_{n+1}\right) + d\left(y_{n+1}, y_{n+2}\right) + ... + d\left(y_{n+k-1}, y_{n+k}\right)$.By (1) this yields

$$d\left(y_n, y_m\right) \leq p^n d\left(y_0, y_1\right) + p^{n+1} d\left(y_0, y_1\right) + ... + p^{n+k-1} d\left(y_0, y_1\right)$$
$$\leq p^n \left(1 + p + p^2 + ... + p^{k-1}\right) d\left(y_0, y_1\right)$$
$$d\left(y_n, y_m\right) \leq p^n \left(\frac{1-p^k}{1-p}\right) d\left(y_0, y_1\right) \leq \frac{p^n}{1-p} d\left(y_0, y_1\right)$$

Since $p < 1$ therefore taking limit $n \to \infty$.This yields $d\left(y_n, y_m\right) \to 0$ and hence $\{y_n\}$ is a Cauchy sequence. So it will be convergent at some point u in X or $\lim fx_n = \lim gx_n = u$. Since pair (f, g) is semi compatible, this yield $\lim fgx_n = gu$. Also g is continuous then $\lim ggx_n = gu$. Now by using (a)

$d\left(fx_n, fgx_n\right) \geq ad\left(fx_n, gx_n\right) + bd\left(fgx_n, ggx_n\right) + cd\left(gx_n, ggx_n\right)$.

Now limiting $n \to \infty$, $d(u, gu) \geq ad(u, u) + bd(gu, gu) + cd(u, gu)$.

Since $c > 1$, this yields $gu = u$. Again by using (a)

$d\left(fu, fx_n\right) \geq ad\left(fu, gu\right) + bd\left(fx_n, gx_n\right) + cd\left(gu, gx_n\right)$. Now limiting $n \to \infty$, we have $d(fu, u) \geq ad(fu, u) + bd(u, u) + cd(u, u)$. Since $a > 1$ this yields $fu = u$. Therefore u is common fixed point of f & g . Let v is another fixed point of f & g . Then by using (a), $d(fu, fv) \geq ad(fu, gu) + bd(fv, gv) + cd(gu, gv)$.

Since $c > 1$ this yields $u = v$, and hence uniqueness proved.

Corollary 2.17- Let (X,d) be a complete metric space and f be a function mapping from X into itself, satisfying the following conditions

(a) $d(fx,fy) \geq ad(fx,x) + bd(fy,y) + cd(x,y)$

Where a,b,c are numbers, satisfy $a>1$,$c>1$, $b \in R$ $(a+b+c>1)$.

Then, f has a unique fixed point in X.

Corollary 2.18- Let (X,d) be a complete metric space and f & g be a function mapping from X into itself, satisfying the following conditions

(a) $d(fx,fy) \geq ad(fx,gx) + bd(fy,gy) + c\min\left[d(fy,gx), d(fx,gy), d(gx,gy)\right]$

Where a,b,c are numbers, satisfy $a>1$,$b \in R(a+b>1)$ & $c>1$.

(b) Pair (f,g) is semi compatible with g is continuous.

Then, f & g have unique common fixed point in X.

3. Main Results-

Theorem 3.1-Let (X,G) be a complete G-metric space and $f,g : X \to X$ are mapping satisfies the following conditions

(a) $G(fx,fy,fy) \geq aG(fx,gx,gx) + bG(fy,gy,gy) + cG(gx,gy,gy)$

Or

(b) $G(fx,fx,fy) \geq aG(fx,fx,gx) + bG(fy,fy,gy) + cG(gx,gx,gy)$

For all $x,y \in X$, where $a>1$,$c>1$ & $b \in R$ $(a+b+c>1)$

If $f(x) \subseteq g(x)$ and pair (f,g) is semi compatible, also g is continuous, then f & g have unique common fixed point in X.

Proof- Suppose that f & g satisfy the condition (a) and (b). If (X,G) is symmetric, then by adding these, we have

$$d_G(fx,fy) \geq \frac{a}{2}d_G(fx,gx) + \frac{b}{2}d_G(fy,gy) + \frac{c}{2}d_G(gx,gy) + \frac{a}{2}d_G(fx,gx) + \frac{b}{2}d_G(fy,gy) + \frac{c}{2}d_G(gx,gy)$$

$$d_G(fx,fy) \geq ad_G(fx,gx) + bd_G(fy,gy) + cd_G(gx,gy)$$

In this inequality since $a+b+c>1$ & $x,y\in X$, the existence and uniqueness of common fixed point follows from theorem (2.16). However If (X,G) is

Non-symmetric then by definition of metric d_G on X and proposition (2.9),

$$d_G(fx,fy)=G(fx,fy,fy)+G(fx,fx,fy)$$

$$d_G(fx,fy)\geq \frac{a}{3}d_G(fx,gx)+\frac{b}{3}d_G(fy,gy)+\frac{c}{3}d_G(gx,gy)$$
$$+\frac{a}{3}d_G(fx,gx)+\frac{b}{3}d_G(fy,gy)+\frac{c}{3}d_G(gx,gy)$$

$d_G(fx,fy)\geq \frac{2a}{3}d_G(fx,gx)+\frac{2b}{3}d_G(fy,gy)+\frac{2c}{3}d_G(gx,gy)$, for all $x,y\in X$, here the expansive factor $\frac{2a}{3}+\frac{2b}{3}+\frac{2c}{3}=\frac{2}{3}(a+b+c)$ need not be less than 1 .Therefore metric d_G gives no information. But the existence of fixed point can be proved by using the properties of G-metric space.

Let x_0 be an arbitrary point in X. We define the sequence $fx_{n+1}=gx_n=y_n$, $n=0,1,2..$ and then condition (a) implies that,

$$G(fx_n,fx_{n+1},fx_{n+1})\geq aG(fx_n,gx_n,gx_n)+bG(fx_{n+1},gx_{n+1},gx_{n+1})+cG(gx_n,gx_{n+1},gx_{n+1})$$
$$G(y_{n-1},y_n,y_n)\geq aG(y_{n-1},y_n,y_n)+bG(y_n,y_{n+1},y_{n+1})+cG(y_n,y_{n+1},y_{n+1})$$

$G(y_n,y_{n+1},y_{n+1})\leq \frac{1-a}{b+c}G(y_{n-1},y_n,y_n)$, Since $\frac{1-a}{b+c}<1$ or $a+b+c>1$. Let $q=\frac{1-a}{b+c}$ then $G(y_n,y_{n+1},y_{n+1})\leq qG(y_{n-1},y_n,y_n)$.Continuing in same argument we will have

$G(y_n,y_{n+1},y_{n+1})\leq q^nG(y_0,y_1,y_1)$. By lemma (2.13), $\{y_n\}$ is a $G-$Cauchy sequence, then by completeness of (X,G), there exist $u\in X$ such that $\{y_n\}$ is G-convergent to u.

Consequently $\lim fx_n=u$ & $\lim gx_n=u$. Since pair (f,g) is semi compatible then

$\lim G(fgx_n,gu,gu)=0\Rightarrow \lim fgx_n=gu$.Also g is continuous then $\lim ggx_n=gu$.

Now by using (a),

$$G(fgx_n,fx_n,fx_n)\geq aG(fgx_n,ggx_n,ggx_n)+bG(fx_n,gx_n,gx_n)+cG(ggx_n,gx_n,gx_n)$$

Now limiting $n\to\infty$ yields $G(gu,u,u)\geq aG(gu,gu,gu)+bG(u,u,u)+cG(gu,u,u)$

Since $c>1$ this yields $gu=u$. Again by using (a)

$G(fu, fx_n, fx_n) \geq aG(fu, gu, gu) + bG(fx_n, gx_n, gx_n) + cG(gu, gx_n, gx_n)$.

Now limiting $n \to \infty$ yields $G(fu,u,u) \geq aG(fu,u,u) + bG(u,u,u) + cG(u,u,u)$. Since $a > 1$, yields $fu = u$. This shows that u is common fixed point of f and g.

Uniqueness-Let v be another fixed point of f & g then by (a),

$G(fu, fv, fv) \geq aG(fu, gu, gu) + bG(fv, gv, gv) + cG(gu, gv, gv)$

$G(u,v,v) \geq aG(u,u,u) + bG(v,v,v) + cG(u,v,v)$.Since $c > 1$ this yields $u = v$.

If f & g satisfy condition (b) then the argument is similar to the above. However to show that sequence $\{y_n\}$ is $G-$Cauchy. By using (b),

$G(fx_n, fx_n, fx_{n+1}) \geq aG(fx_n, fx_n, gx_n) + bG(fx_{n+1}, fx_{n+1}, gx_{n+1}) + cG(gx_n, gx_n, gx_{n+1})$
$G(y_{n-1}, y_{n-1}, y_n) \geq aG(y_{n-1}, y_{n-1}, y_n) + bG(y_n, y_n, y_{n+1}) + cG(y_n, y_n, y_{n+1})$

$G(y_n, y_n, y_{n+1}) \leq \frac{1-a}{b+c} G(y_{n-1}, y_{n-1}, y_n)$. Since $\frac{1-a}{b+c} < 1$. Let $\frac{1-a}{b+c} = q$, then

$G(y_n, y_n, y_{n+1}) \leq qG(y_{n-1}, y_{n-1}, y_n)$. Continuing in same argument we have

$G(y_n, y_n, y_{n+1}) \leq q^n G(y_0, y_0, y_1)$. By lemma (2.13) $\{y_n\}$ is G Cauchy sequence.

Corollary3.2- Let (X,G) be a complete G-metric space and $f, g : X \to X$ be a mapping satisfies the following condition,

(a) $G(f^m x, f^m y, f^m y) \geq aG(f^m x, g^m x, g^m x) + bG(f^m y, g^m y, g^m y) + cG(g^m x, g^m y, g^m y)$
Or
(b) $G(f^m x, f^m x, f^m y) \geq aG(f^m x, f^m x, g^m x) + bG(f^m y, f^m y, g^m y) + cG(g^m x, g^m x, g^m y)$

For all $x, y \in X$, where $a > 1$, $c > 1$ & $b \in R$ $(a + b + c > 1)$

If $f(x) \subseteq g(x)$ and pair (f, g) is semi compatible, also g is continuous, then f & g have unique common fixed point in X.

Proof- From the previous theorem we see that f^m & g^m have unique fixed point (say u), that is $f^m(u) = u$ & $g^m(u) = u$. But $f(u) = f(f^m(u)) = f^m(f(u))$, therefore $f(u)$ is another fixed point of f^m. And by uniqueness $f(u) = u$. By similar argument that $g(u) = u$.Therefore u is unique common fixed point of f & g.

If we take g is an identity map in theorem (3.1) we get following corollary

Corollary 3.3- Let (X,G) be a complete G-metric space and $f:X\to X$ be a mapping satisfies the following condition

(a) $G(fx,fy,fy)\geq aG(fx,x,x)+bG(fy,y,y)+cG(x,y,y)$

Or

(b) $G(fx,fx,fy)\geq aG(fx,fx,x)+bG(fy,fy,y)+cG(x,x,y)$

For all $x,y\in X$, where $a>1$, $c>1$ & $b\in R$ $(a+b+c>1)$

Then f has unique fixed point in X .

Proof- This will follow theorem (3.1) and can be proved with the help of corollary (2.17).

Corollary3.4- Let (X,G) be a complete G-metric space and $f:X\to X$ be a mapping satisfies the following condition

(a) $G\left(f^m x,f^m y,f^m y\right)\geq aG\left(f^m x,x,x\right)+bG\left(f^m y,y,y\right)+cG(x,y,y)$

Or

(b) $G\left(f^m x,f^m x,f^m y\right)\geq aG\left(f^m x,f^m x,x\right)+bG\left(f^m y,f^m y,y\right)+cG(x,x,y)$

For all $x,y\in X$, where $a>1$, $c>1$ & $b\in R$ $(a+b+c>1)$

Then f^m has unique fixed point in X .

Proof- From the previous corollary we see that f^m has unique fixed point (say u), that is $f^m(u)=u$. But $f(u)=f\left(f^m(u)\right)=f^m\left(f(u)\right)$, therefore $f(u)$ is another fixed point of f^m . But by uniqueness $f(u)=u$.

Theorem 3.5-Let (X,G) be a complete G-metric space and $f,g:X\to X$ are mapping satisfies the following conditions

$$(a)\ G(fx,fy,fy)\geq aG(fx,gx,gx)+bG(fy,gy,gy)$$
$$+c\min\left[G(fy,gx,gx),G(fx,gy,gy),G(gx,gy,gy)\right]$$

Or

$$(b)\ G(fx,fx,fy)\geq aG(fx,fx,gx)+bG(fy,fy,gy)$$
$$+c\min\left[G(fy,fy,gx),G(fx,fx,gy),G(gx,gx,gy)\right]$$

For all $x,y\in X$, where $a>1,b\in R(a+b>1)$ & $c>2$

If $f(x) \subseteq g(x)$ and pair (f,g) is semi compatible, also g is continuous, then f & g have unique common fixed point in X .

Proof- Suppose that f & g satisfy the condition (a) and (b). If (X,G) is symmetric, then by adding these, we have

$$d_G(fx, fy) \geq \frac{a}{2} d_G(fx, gx) + \frac{b}{2} d_G(fy, gy) + \frac{c}{2} \min\left[d_G(fy, gx), d_G(fx, gy), d_G(gx, gy)\right]$$
$$+ \frac{a}{2} d_G(fx, gx) + \frac{b}{2} d_G(fy, gy) + \frac{c}{2} \min\left[d_G(fy, gx), d_G(fx, gy), d_G(gx, gy)\right]$$
$$d_G(fx, fy) \geq a d_G(fx, gx) + b d_G(fy, gy) + c \min\left[d_G(fy, gx), d_G(fx, gy), d_G(gx, gy)\right]$$

In this inequality since $a+b+c>1$ & $x, y \in X$, the existence and uniqueness of common fixed point follows from corollary (2.18). However If (X,G) is non-symmetric then by definition of metric d_G on X and proposition (2.9),

$$d_G(fx, fy) = G(fx, fy, fy) + G(fx, fx, fy)$$
$$d_G(fx, fy) \geq \frac{a}{3} d_G(fx, gx) + \frac{b}{3} d_G(fy, gy) + \frac{c}{3} \min\left[d_G(fy, gx), d_G(fx, gy), d_G(gx, gy)\right]$$
$$+ \frac{a}{3} d_G(fx, gx) + \frac{b}{3} d_G(fy, gy) + \frac{c}{3} \min\left[d_G(fy, gx), d_G(fx, gy), d_G(gx, gy)\right]$$
$$d_G(fx, fy) \geq \frac{2a}{3} d_G(fx, gx) + \frac{2b}{3} d_G(fy, gy) + \frac{2c}{3} \min\left[d_G(fy, gx), d_G(fx, gy), d_G(gx, gy)\right]$$

For all $x, y \in X$, here the expansive factor $\frac{2a}{3} + \frac{2b}{3} + \frac{2c}{3} = \frac{2}{3}(a+b+c)$ need not be less than 1.Therefore metric d_G gives no information. But the existence of fixed point can be proved by using the properties of G -metric space.

Let x_0 be an arbitrary point in X . We define the sequence $fx_{n+1} = gx_n = y_n$, $n = 0,1,2..$ and then condition (a) implies that,

$$G(fx_n, fx_{n+1}, fx_{n+1}) \geq aG(fx_n, gx_n, gx_n) + bG(fx_{n+1}, gx_{n+1}, gx_{n+1})$$
$$+ c \min\left[G(fx_{n+1}, gx_n, gx_n), G(fx_n, gx_{n+1}, gx_{n+1}), G(gx_n, gx_{n+1}, gx_{n+1})\right]$$
$$G(y_{n-1}, y_n, y_n) \geq aG(y_{n-1}, y_n, y_n) + bG(y_n, y_{n+1}, y_{n+1})$$
$$+ c \min\left[G(y_n, y_n, y_n), G(y_{n-1}, y_{n+1}, y_{n+1}), G(y_n, y_{n+1}, y_{n+1})\right]$$

$G(y_n, y_{n+1}, y_{n+1}) \le \frac{1-a}{b} G(y_{n-1}, y_n, y_n)$, Since $\frac{1-a}{b} < 1 \; or \; a+b>1$. Let $q = \frac{1-a}{b}$ then $G(y_n, y_{n+1}, y_{n+1}) \le qG(y_{n-1}, y_n, y_n)$.Continuing in same argument we will have $G(y_n, y_{n+1}, y_{n+1}) \le q^n G(y_0, y_1, y_1)$. By lemma (2.13), $\{y_n\}$ is a $G-$Cauchy sequence, then by completeness of (X, G), there exist $u \in X$ such that $\{y_n\}$ is G-convergent to u.

Consequently $\lim fx_n = u$ & $\lim gx_n = u$. Since pair (f, g) is semi compatible then $\lim G(fgx_n, gu, gu) = 0 \Rightarrow \lim fgx_n = gu$ .Also g is continuous then $\lim ggx_n = gu$.

Now by using (a),

$$G(fgx_n, fx_n, fx_n) \ge aG(fgx_n, ggx_n, ggx_n) + bG(fx_n, gx_n, gx_n) + c\min\left[G(fx_n, ggx_n, ggx_n), G(fgx_n, gx_n, gx_n), G(ggx_n, gx_n, gx_n)\right]$$

Now limiting $n \to \infty$ we get,

$$G(gu,u,u) \ge aG(gu, gu, gu) + bG(u,u,u) + c\min\left[G(u, gu, gu), G(gu,u,u), G(gu,u,u)\right]$$

By proposition (2.2) it can be easily obtained that

$$G(gu,u,u) \ge c\min\left[\tfrac{1}{2}G(gu,u,u), G(gu,u,u), G(gu,u,u)\right]$$

$G(gu,u,u) \ge \frac{c}{2} G(gu,u,u)$. Since $c > 2$ this yields $gu = u$. Again by using (a)

$$G(fu, fx_n, fx_n) \ge aG(fu, gu, gu) + bG(fx_n, gx_n, gx_n) + c\min\left[G(fx_n, gu, gu), G(fu, gx_n, gx_n), G(gu, gx_n, gx_n)\right]$$

Now limiting $n \to \infty$

$$G(fu,u,u) \ge aG(fu,u,u) + bG(u,u,u) + c\min\left[G(u,u,u), G(fu,u,u), G(u,u,u)\right]$$

$G(fu,u,u) \ge aG(fu,u,u)$. Since $a > 1$, this yields $fu = u$. Therefore u is common fixed point of f and g .

Uniqueness can be easily proved for this theorem.

If f & g satisfy condition (b) then the argument is similar to the above theorem. However to show that sequence $\{y_n\}$ is G-Cauchy. By using (b),

$$G(fx_n, fx_n, fx_{n+1}) \ge aG(fx_n, fx_n, gx_n) + bG(fx_{n+1}, fx_{n+1}, gx_{n+1}) + c\min\left[G(fx_{n+1}, fx_{n+1}, gx_n), G(fx_n, fx_n, gx_{n+1}), G(gx_n, gx_n, gx_{n+1})\right]$$

$$G(y_{n-1},y_{n-1},y_n)\geq aG(y_{n-1},y_{n-1},y_n)+bG(y_n,y_n,y_{n+1})$$
$$+c\min\left[G(y_n,y_n,y_n),G(y_{n-1},y_{n-1},y_{n+1}),G(y_n,y_n,y_{n+1})\right]$$

$G(y_n,y_n,y_{n+1})\leq \frac{1-a}{b}G(y_{n-1},y_{n-1},y_n)$, Since $\frac{1-a}{b}<1$ or $a+b>1$. Let $q=\frac{1-a}{b}$ then $G(y_n,y_n,y_{n+1})\leq qG(y_{n-1},y_{n-1},y_n)$. Continuing in same argument we will have $G(y_n,y_n,y_{n+1})\leq q^nG(y_0,y_0,y_1)$. By lemma (2.13), $\{y_n\}$ is a $G-$Cauchy sequence.

Corollary3.6- Let (X,G) be a complete G -metric space and $f,g:X\to X$ be a mapping satisfies the following condition,

$$G(f^m x,f^m y,f^m y)\geq aG(f^m x,g^m x,g^m x)+bG(f^m y,g^m y,g^m y)$$
$$+c\min\left[G(f^m y,g^m x,g^m x),G(f^m x,g^m y,g^m y),G(g^m x,g^m y,g^m y)\right]$$

Or

$$G(f^m x,f^m x,f^m y)\geq aG(f^m x,f^m x,g^m x)+bG(f^m y,f^m y,g^m y)$$
$$+c\min\left[G(f^m y,f^m y,g^m x),G(f^m x,f^m x,g^m y),G(g^m x,g^m x,g^m y)\right]$$

For all $x,y\in X$, where $a>1,b\in R(a+b>1)$ & $c>2$

If $f(x)\subseteq g(x)$ and pair (f,g) is semi compatible, also g is continuous, then f & g have unique common fixed point in X .

Proof- We use the same argument as in corollary (3.2).

Conflict of Interests

The author declares that there is no conflict of interests.

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