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FIXED POINT THEOREMS FOR CONTRACTION AND NONEXPANSIVE MAPPINGS DEFINED ON CARTESIAN PRODUCTS SPACES

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Abstract. In this paper, we study some facts concerning the topological structures of some subsets of a given normed space, prove the existence of a unique fixed point of r-contraction mappings defined on cartesian product of a closed and weakly Cauchy subset of a weakly complete normed space, propose an iterated sequence that converges strongly to such a unique fixed point, and prove the existence of fixed points of nonexpansive mapping defined on cartesian product of a closed and sequentially Cauchy subset of a weakly complete normed space.

Keywords: fixed point; r-contraction; nonexpansive mappings; cartesian product; weakly Cauchy.

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1. Introduction

In 1912, L. E. J. Brouwer [2] proved his famous fixed point theorem, if R^n is the n-Euclidian space with the usual metric and T is a continuous mapping from R^n into R^n, then T has fixed point. In 1922, S. Banach [1] introduced his Banach contraction principle. If X is a complete metric space and T is r-contraction mapping from X into itself, then T has a unique fixed point $y \in X$. Moreover, the sequence of iterates $\{T^n(x)\}_{n\in\mathcal{N}}$ is strongly convergent to y for

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every $x \in X$. Mathematicians in the field of fixed point theory try to improve the results of this theorem in which reduce the contractivity assumption imposed on the given mapping or changing the completeness condition on the given topological space. In 1965, [5] F. Edelstein theorem was introduced to prove that if X is a compact metric space and T is a contractive mapping from X into itself, then T has a unique fixed point $y \in X$. Moreover, the sequence of iterates $\{T^n(x)\}_{n\in\mathcal{N}}$ is strongly convergent to y for every $x \in X$. In 1976, C. S. Wong [9] proved that if X is a uniformly convex Banach space C compact convex subset of X, T is a generalized nonexpansive mapping from C into C, then T has fixed point $x \in C$. In 1984, K. Geobel and S. Riech [4] proved that if X is a uniformly convex Banach space C closed bounded convex subset of X, T is a nonexpansive mapping on C, then T has fixed point $x \in C$. Many research papers have been written to generalize the fixed point theorems for the contractions and the nonexpansive mappings. The nonexpansive mappings in general may not have fixed points even on uniformly convex Banach spaces and contractive mappings require compact metric spaces. On the other hand, T. G. Bhaskar and V. Lakshmikanthan [3] in 2006 considered a mixed monotone mapping in a Banach space endowed with partial order, using a weak contractivity type assumption to prove the following theorem:

Theorem 1.1. *Let X be a sequentially lower-upper ordered Banach space $T : X \times X \to X$ be a mapping having the first-anti-second and first-second mixed monotone properties and satisfy the following condition:*

$$\|T(x,y) - T(u,v)\| \leq \frac{r}{2}[\|x-u\| + \|y-v\|]$$

for all x, y, u, and v in X with $x \geq u$, and $y \leq v$, where $0 \leq k < 1$. *Then T has a first-anti-second couple fixed point in X.*

In 2011, the first author [7] introduced the concept of a Right and Left-Double $\{a,b,c\}$ type contraction mappings from $C \times C$ into C, where C is a closed weakly Cauchy normed subspace of a normed subspace X and proved the existence of a fixed point theorems of such types of mappings.

In this paper, we study the contractions and nonexpansive mappings on cartesian products spaces

spaces $C\times C$ into C and prove the existence of a fixed points of mappings defined from the Cartesian product $C\times C$ to C, where C is some sequentially Cauchy and weakly Cauchy normed subspace of a given normed space X.

2. Preliminaries

First, we have the following definitions.

Definition 2.1. Let X be a normed space and C be a subset of X. Then

(1) A function $f : X \to (-\infty,\infty]$ is said to be proper lower semicontinuous and convex function if it satisfies the following conditions [5], [8]:
- There is $x\in X$ such that $f(x)<\infty$ (proper).
- For any real number α, the set $\{x\in X : f(x)\leq \alpha\}$ is closed convex subset of X (lower semicontinuous).
- For any $x,y\in X$ and $t\in[0, 1]$, $f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)$(convex function).

(2) C is said to be sequentially Cauchy if and only if every sequence in C has a Cauchy subsequence.

(3) C is said to be weakly Cauchy subset of X if and only if every Cauchy sequence in C converges weakly to some element x in X; [6].

(4) A normed space X is said to be weakly complete if and only if every Cauchy sequence in X is converging weakly to some point x in X.

Remark 2.1.

(1) Let X be a normed space, C be a closed subset of X, $\{x_n\}_{n\in\mathscr{N}}$ be a Cauchy sequence in C that is weakly convergent to some point x_0 in X. Then $\{x_n\}_{n\in\mathscr{N}}$ converges strongly to x_0 and $x_0\in C$. In fact; if Y is the strong completion of X, then there is an element $y\in Y$ such that $\{x_n\}_{n\in\mathscr{N}}$ converges strongly to y, since the strong convergent implies the weak convergence to the same strong limit, the sequence $\{x_n\}_{n\in\mathscr{N}}$ will converge weakly to y, using the fact of the unique limit of a sequence proves that $y=x_0$, therefore

the weak convergence is in fact a strong convergence. On the other side $x \in C$ because C is closed in X.

(2) A closed subset of a weakly complete normed space is complete.

(3) Every totally bounded (hence every compact) subset of a normed space is sequentially Cauchy.

(4) Every subset of a sequentially Cauchy is sequentially Cauchy.

(5) Every sequentially Cauchy is bounded.

(6) C is complete if and only if it is both closed and weakly Cauchy.

(7) C is compact (sequentially compact) if and only if it is both sequentially Cauchy and complete.

(8) C is compact if and only if it is closed, sequentially Cauchy, and weakly Cauchy.

We have the following examples.

Example 2.1. Let R be the normed space of real numbers with the absolute value metric and X be the semi open or (closed) interval $X = [a,d)$ in R. Then

(1) X is totally bounded in R hence it is sequentially Cauchy while it is not compact in R.

(2) If b is a real number $a < b < d$ and $C = [b,d)$, then
   - C is closed in X because it contains all its limit points from X.
   - C is not closed in the completion of X, because d is a limit point of C in $[a,d]$ while C does not contain d.
   - C is not weakly Cauchy in X, because $\{(\frac{1}{n})a + (1-\frac{1}{n})d\}_{n\in N}$ is a Cauchy (convergent) sequence in C with no weak limit in X.
   - C is sequentially Cauchy in X as well as it is in R.

The p-cartesian product norm on $C \times C$ is the norm induced on $C \times C$ by the norm given on C, it may be one of the following norms:

$$\|(x,y)\|_p = \begin{cases} [\|x\|^p + \|y\|^p]^{\frac{1}{p}}, if\ 1 \leq p < \infty; \\ Max\{\|x\|, \|y\|\}, if\ p = \infty. \end{cases}$$

Definition 2.2. Let X and Y be two metric spaces with metrics d_X and d_Y respectively and T be a mapping from X into Y. Then

(1) T is said to be r-Lipschitzian where r is a nonnegative real number if and only if

$$d_Y(T(x), T(y)) \leq r d_X(x, y) \text{ for every x and y in X.}$$

(2) T is said to be contraction or r-contraction if and only if r is a nonnegative real number r, $0 \leq r < 1$ such that

$$d_Y(T(x), T(y)) \leq r d_X(x, y) \text{ for every x and y in X.}$$

(3) T is said to be nonexpansive if and only if

$$d_Y(T(x), T(y)) \leq d_X(x, y) \text{ for every x and y in X.}$$

(4) T is said to be contractive if and only if

$$d_Y(T(x), T(y)) < d_X(x, y) \text{ for every x, y in X with } x \neq y.$$

In the case of normed spaces, we replace $d(x, y)$ by simply $\|x - y\|$.

Definition 2.3. Let X be a normed space, C be a subset of X and T be a mapping from $C \times C$ into C. Then

(1) The point $x_0 \in C$ is said to be fixed point of T if and only if $T((x_0, x_0)) = x_0$.

(2)
- T is said to be diagonally r-Lipschitzian where r is a nonnegative real number if and only if

$$\|T((x, x)) - T((y, y))\| \leq r\|x - y\| \text{ for every x and y in C.}$$

- T is said to be diagonally r-contraction if and only if there r is a nonnegative real number r, $0 \leq r < 1$ such that

$$\|T((x, x)) - T((y, y))\| \leq r\|x - y\| \text{ for every x and y in C.}$$

- T is said to be nonexpansive if and only if

$$\|T((x, x)) - T((y, y))\| \leq \|x - y\| \text{ for every x and y in C.}$$

Example 2.2.

- Consider the mapping $T : R \times R \to R$ that is defined by

$$T((x,y)) = (y-1)(x+3) + xy + (x-1)(y^2+6)$$

  such a mapping has a unique fixed point $x_0 = 1$ in R.

- Consider the mapping $T : R \times R \to R$ that is defined by

$$T((x,y)) = y\, sin(x) + x\, sin^2(y) + y$$

  such a mapping has an infinitely many set of fixed points

$$Fix(T) = \{[\frac{3+4n}{2}]\pi : n \in Z\} \bigcup \{n\pi : n \in Z\}.$$

- Consider the mapping $T : R \times R \to R$ that is defined by

$$T((x,y)) = 2y - x$$

  such a mapping has an infinitely many set of fixed points $Fix(T) = R$.

- Consider the mapping $T : R \times R \to R$ that is defined by

$$T((x,y)) = y^2 + 1$$

  such a mapping has no fixed point or fixed point free mapping $Fix(T) = \phi$.

- Consider the two mappings $T_{1,2} : R \times R \to R$ that are defined by

$$T_1((x,y)) = (\frac{1}{2})[x^2 + (y-1)(y^2+2y-1) + x + y - 1]$$

  and by

$$T_2((x,y)) = x^2 + (y-1)(y^2+2y-1) + y - 1$$

  such two mappings have the same set of fixed points $Fix(T) = \{0, 1, -3\}$.

We noticed the following:

Remark 2.2.

(1) Let T be a mapping from $C \times C$ into C and $z \in C$ be an arbitrarily element in C, define the right and left mappings T_{rz} and T_{lz} from C into C as follows:

$$T_{rz}(x) = T((x,z)) \text{ and } T_{lz}(x) = T((z,x)) \text{ for every x in C.}$$

Then

- If x is a common fixed point of the family of mappings $C_1 := \{T_{rz} : z \in C\}$ or $C_2 := \{T_{lz} : z \in C\}$, then x is a fixed point of T, $\bigcap\{Fix(T_{rz}) : z \in C\} \subset Fix(T)$ and $\bigcap\{Fix(T_{lz}) : z \in C\} \subset Fix(T)$.
- If x is a fixed point of T, then x is a fixed point of at least one element in $C_1 := \{T_{rz} : z \in C\}$ and one element in $C_2 := \{T_{lz} : z \in C\}$.
- If T is r-contraction mapping from $C \times C$ into C, then each member of the families $C_1 := \{T_{rz} : z \in C\}$ and $C_2 := \{T_{lz} : z \in C\}$ is r-contraction, because

$$\begin{aligned} \|T_{rz}(x) - T_{rz}(y)\| &= \|T((x,z)) - T((y,z))\| \\ &\leq r\|(x,z) - (y,z)\|_p \\ &= r\|(x-y,0)\|_p = r\|x-y\| \end{aligned}$$

  for every x and y in C.

(2) Every r-contraction mapping from $C \times C$ into C is diagonally $[2^{\frac{1}{p}} r]$-Lipschetzian. In fact; the inequality

$$\|T((x,u)) - T((y,v))\| \leq r\|(x-y, u-v)\|_p \text{ for every x, u, y and v in C.}$$

implies easily the inequality (simply take $x = u$ and $y = v$)

$$\|T((x,x)) - T((y,y))\| \leq [2^{\frac{1}{p}} r]\|x-y\| \text{ for every x and y in C.}$$

(3) The mapping T from $C \times C$ into C is diagonally r-contraction if and only if the induced mapping T_C defined from C into C by $T_C(x) = T((x,x))$ is r-contraction and the point x is a fixed point of T if and only if x is a fixed point of T_C.

(4) If $\triangle(C) = \{(x,x) : x \in C\}$ and T is a mapping from C into C, then T is r-contraction if and only if the induced mapping $T_{\triangle(C)}$ defined from $\triangle(C)$ into C by $T_{\triangle(C)}((x,x)) = T(x)$ is diagonally r-contraction and the point x is a fixed point of T if and only if x is a fixed point of $T_{\triangle(C)}$.

Definition 2.4. Let X be a normed space, C be a subset of X, and T be a mapping from $C\times C$ into C. If x is arbitrarily element in C, then the sequence $\{x_n\}_{n\in N}$ defined iteratively by

$$x_1 = T(x,x), x_2 = T(x_1,x_1), \ldots x_n = T(x_{n-1},x_{n-1}), \ldots \tag{1}$$

will be denoted by $x_n = T^n(x,x)$.

3. Main results

The following lemma is easy to prove. So, we omit the proof.

Lemma 3.1. *Let X be a weakly complete normed space and $\{C_n\}_{n\in N}$ be a descending sequence of closed and sequentially Cauchy subsets of X. Then the intersection $\bigcap_{n\in N} C_n$ is nonempty set*

$$\bigcap_{n\in N} C_n \neq \phi$$

Now, we are in a position to prove our main result.

Theorem 3.2. *Let X be a weakly complete normed space, C be a closed, convex, and sequentially Cauchy subset of X, and $\{x_n\}_{n\in N}$ be a sequence in C. Then the proper, convex, lower semi continuous, and nonnegative real valued function φ on C defined by*

$$\varphi(x) := \limsup_{n\to\infty} \|x_n - x\|, x\in C$$

attains its minimum at exactly one element in C.

Proof. Denote by d, $d = inf\{\varphi(x) : x\in C\}$ and let $\{\alpha_n\}_{n\in N}$ be a sequence of positive real numbers such that $\lim_{n\to\infty}\alpha_n = 0$. Then define the sequence of descending closed and sequentially Cauchy subsets of C as follows:

$$C_n := \{x : x\in C, d\le \varphi(x) < d+\alpha_n\}.$$

Using lemma 3.1, we prove the existence of a point $x_0\in C$ such that

$$x_0 \in \bigcap_{n\in N} C_n,$$

hence $d \leq \varphi(x_0) < d + \alpha_n \forall n \in N$, from which we see that $\varphi(x_0) = d$, that is; there is $x_0 \in C$ such that $\varphi(x_0) = \min\{\varphi(x) : x \in C\}$. To prove that such element is unique we notice that $\varphi(\frac{x+y}{2}) < \max\{\varphi(x), \varphi(y)\}$ for every x, y and $x \neq y$ in C. Now; suppose that there are two distinct elements x, y in C such that $\varphi(x) = \varphi(y) = \min\{\varphi(z) : z \in C\}$, since C is convex we see that $\frac{x+y}{2} \in C$ and the following strict inequality is an obvious contradiction:

$$\min\{\varphi(z) : z \in C\} \leq \varphi(\frac{x+y}{2}) < \max\{\varphi(x), \varphi(y)\} = \varphi(x) = \varphi(y) = \min\{\varphi(z) : z \in C\}.$$

Lemma 3.3. *Let X be a normed space, C be a subset of X, and T be a diagonally r-contraction mapping from $C \times C$ into C. If x is an arbitrarily element in C, then the sequence $\{x_n\}_{n\in N}$ defined iteratively as in (1) satisfying the following:*

$$\|T^{n+1}(x,x) - T^n(x,x)\| \leq r^n \|T^2(x,x) - T(x,x)\| \tag{2}$$

for any natural numbers n, m and $n \leq m$ we have

$$\|T^m(x,x) - T^n(x,x)\| \leq [\frac{r^n}{1-r}] \|T^2(x,x) - T(x,x)\|. \tag{3}$$

Proof. Using the induced mapping $T_C(x) = T((x,x))$ with the same steps of the Banach contraction principle completes the proof.

Theorem 3.4. *Let X be a weakly complete normed space, C be a closed subset of X, and T be a diagonally r-contraction mapping from $C \times C$ into C. If x is an arbitrarily element in C, then T has a unique fixed point. Moreover, the sequence $\{x_n\}_{n\in N}$ defined iteratively as in (1) is strongly convergent to the unique fixed point of T.*

Proof. Since C is closed subset of a weakly complete normed space X, C is complete, the induced mapping is now r-contraction from a Banach space C into itself, using Lemma 3.3 with the same steps of the Banach contraction principle completes the proof.

Theorem 3.5. *Let X be a weakly complete normed space, C be closed, convex, and sequentially Cauchy subset of X, and T be a diagonally nonexpansive mapping from $C \times C$ into C. Then there is a sequence $\{x_n\}_{n\in N}$ in C such that $\lim_{n\to\infty} \|T((x_n,x_n)) - x_n\| = 0$. Hence proves that*

$$\inf\{\|T((x,x)) - x\| : x \in C\} = 0.$$

Moreover, T has fixed points.

Proof. Consider the induced mapping $T_C(x) = T((x,x))$, let $\{r_n\}_{n\in N}$ be a sequence of positive real numbers such that $0 \le r_n < 1$ and $\lim_{n\to\infty} r_n = 1$ and z be arbitrarily element in C . Then define the sequence of r_n-contraction mappings from C into C as follows:

$$S_n(x) = (1-r_n)z + r_n T_C(x).$$

If $\{x_n\}_{n\in N}$ is a sequence of unique fixed points, each x_n is the unique fixed point of S_n, then we have the following:

$$\begin{aligned}\|T((x_n,x_n)) - x_n\| &= \|T((x_n,x_n)) - [(1-r_n)z + r_n T_C(x_n)]\| \\ &= \|T((x_n,x_n)) - [(1-r_n)z + r_n T((x_n,x_n))]\| \\ &= (1-r_n)\|T((x_n,x_n)) - z\| \le (1-r_n)Diameter(C).\end{aligned}$$

Taking the limit as $n \to \infty$ proves that

$$\lim_{n\to\infty} \|T((x_n,x_n)) - x_n\| = 0. \tag{4}$$

Using Theorem 3.4 proves the existence of a unique element $x_0 \in C$ such that

$$\varphi(x_0) := \min\{\varphi(z) : z \in C\}. \tag{5}$$

Now, we have $\|x_n - T((x_0,x_0))\| \le \|x_n - T((x_n,x_n))\| + \|T((x_n,x_n)) - T((x_0,x_0))\| \le \|x_n - T((x_n,x_n))\| + \|x_n - x_0\|$ hence

$$\limsup_{n\to\infty} \|x_n - T((x_0,x_0))\| \le \limsup_{n\to\infty} \|x_n - T((x_n,x_n))\| + \limsup_{n\to\infty} \|x_n - x_0\|.$$

Using equation (4) proves that

$$\limsup_{n\to\infty} \|x_n - T((x_0,x_0))\| \le \limsup_{n\to\infty} \|x_n - x_0\|.$$

Equivalently, we have

$$\varphi(T((x_0,x_0))) \le \varphi(x_0). \tag{6}$$

The inequality (6) proves that $T((x_0,x_0))$ is another minimizer for the function φ. using the uniqueness of such minimizer as given in (3.2) proves that $T((x_0,x_0)) = x_0$, therefore x_0 is a fixed point of T.

4. Conclusion

In this paper, we study some facts concerning the topological structures of some subsets of a given normed space. On the other side, let X be a weakly complete normed space, If C is a closed and weakly Cauchy subset of X and T be r-contraction mapping from $C \times C$ into C, then we prove the existence of a unique fixed point of T in the sense that there is a point $x_0 \in C$ such that $T(x_0, x_0) = x_0$, we also propose an iterated sequence in C that converges strongly to such a unique fixed point, and if C is closed and sequentially Cauchy subset of X and T is a nonexpansive mapping from $C \times C$ into C, then we prove the existence of fixed points of T. The results of this paper are new defined.

Conflict of Interests

The authors declare that there is no conflict of interests.

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