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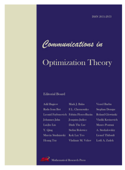

RATIONAL METRIC DIMENSION OF GRAPHS

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Abstract. Let $G=(V,E)$ be a simple connected graph and $S=\{s_1,s_2,\dots s_k\}$ be an ordered subset of V. Then for $u \in V$, we associate a vector $\Gamma(u/S)=(d(u/s_1),d(u/s_2),\dots,d(u/s_k))$ with respect to S, where $d(u/v)=\frac{\sum_{u_i\in N[u]} d(u_i,v)}{deg(u)+1}$. A subset S is said to be rational resolving set if $\Gamma(u/S)\neq\Gamma(v/S)$ for all $u,v\in V-S$. The minimum cardinality of a rational resolving set S is called rational metric dimension and denoted by $rmd(G)$. A rational resolving set S with minimum cardinality is called rational metric basis. In this paper, we compute rational metric dimension of standard graphs and hence show that rational metric basis serves the same purpose of metric basis with fewer vertices than in any classes of metric basis.

Keywords: metric Dimension; landmarks; rational-metric dimension set.

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1. Introduction

All the graphs considered in this paper are simple, connected and undirected. For any two vertices x and y, the distance between two vertices x and y in G, denoted by $d(x,y)$, is the length

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of a shortest path between x and y. A subset S of the vertex set V of a connected graph G is said to be *r*esolving set of G if for every pair of vertices $u,v \in V - S$ there exists a vertex $w \in S$ such that $d(u,w) \neq d(v,w)$. The metric dimension of a graph G, denoted by $\beta(G)$, is the minimum cardinality of a resolving sets S of G. Metric dimension is defined independently by F. Harary et al [3] and P.J. Slater [5] . The terms not defined here may be found in [4]. For the similar work on metric dimension we refer [2, 3, 6-10].

2. Rational Metric dimension

In many practical situation of a network, the role of a vertex is dependent on its neighbourhood. The degree of a vertex plays an important role. Therefore to determine the position of a vertex in the network we need to select landmarks considering not only the distance of the vertex from the land mark but also distances of its neighbourhood vertices from the land marks. In such situations a vertex is viewed as bunch of vertices and an average distance from the land mark is considered. This average distance will be a rational number. Hence we define *Rational Metric Dimension* of graphs which serves all purpose of metric dimension.

Definition 2.1. Let $G = (V,E)$ be a simple connected graph and $u,v \in V$. Then we define

$$d(u/v) = \frac{\sum_{u_i \in N[u]} d(u_i,v)}{deg(u)+1} \tag{1}$$

Definition 2.2. Let $G = (V,E)$ be a simple connected graph and $S = \{s_1, s_2, \ldots s_k\}$ be an ordered subset of V. Then for $u \in V$, we associate a vector $\Gamma(u/S) = (d(u/s_1), d(u/s_2), \ldots, d(u/s_k))$ with respect to S. The set S is said to be rational resolving set if $\Gamma(u/S) \neq \Gamma(v/S)$ for any $u,v \in V - S$. The minimum cardinality of a rational resolving set S is called rational metric dimension and denoted by $rmd(G)$. A rational resolving set S with minimum cardinality is called rational metric basis.

Remark 2.3. For any connected graph $G = (V,E)$;

- If S is a rational resolving set of G, then for every pair of vertices u,v in $V - S$, there exists $w \in S$, such that $d(u/w) \neq d(v/w)$.
- If $w \in S$ and for $u,v \in V - S$, if $d(u/w) \neq d(v/w)$, then we say that w resolves the pair of vertices u and v or w is a resolving vertex for the vertices u and v.

- For any graph G, $1 \leq rmd(G) \leq n-1$
- In a complete graph K_n, $N[v] = V$ for every vertex $v \in V$, hence for any positive integer $n > 1$, $rmd(K_n) = n-1$

3. RMD of some standard graphs

In this section we determine rational metric dimensions of paths, cycles, stars, complete bipartite graphs etc.

Theorem 3.1. For any positive integer n, $rmd(P_n) = 1$

proof: By the definition of $rmd(P_n) \geq 1$. Let $\{v_1, v_2, \ldots v_n\}$ be the vertices of P_n, such that v_i is adjacent to v_{i+1} for each $i = 1, 2, \ldots n-1$ and $S = \{v_1\}$. Then,

$$d(v_i/v_1) = \begin{cases} \frac{1}{2} & for\ i = 1 \\ i & \text{for } 2 \leq i \leq n-1 \\ \frac{2n-1}{2} & \text{for } i = n \end{cases}$$

Thus $\Gamma(v_i/S) \neq \Gamma(v_j/S)$ for every $i \neq j$ and hence S is a rational resolving set of P_n and $rmd(P_n) = 1$.

Theorem 3.2. For any positive integer n, $rmd(C_n) = 2$

proof: Let $\{v_1, v_2, \ldots v_n\}$ be the vertices of C_n, such that v_i is adjacent to v_{i+1} for each $i = 1, 2, \ldots n-1$ and v_n is adjacent to v_1. Let $S = \{v_i : \text{for some } i,\ 1 \leq i \leq n\}$. Then $d(v_{i-1}/v_i) = d(v_{i+1}/v_i)$, hence S is not a rational resolving set of C_n. Thus if S is a rational resolving set of C_n, then $|S| \geq 2$, that is $rmd(C_n) \geq 2$.

i	3	4	5	- - -	$k-1$	k	$k+1$	$k+2$	$k+3$	- - -	$n-1$	n
$d(v_i/v_1)$	2	3	4	- - -	$k-2$	$\frac{3k-1}{3}$	$\frac{3k-1}{3}$	$k-2$	$k-3$	- - -	2	1
$d(v_i/v_2)$	1	2	3	- - -	$k-3$	$k-2$	$\frac{3k-1}{3}$	$\frac{3k-1}{3}$	$k-2$	- - -	3	2

TABLE 1. Values of $d(v_i/v_1)$ and $d(v_i/v_2)$ when $n = 2k+1$

Let $S = \{v_1, v_2\}$. We prove S is a rational resolving set of C_n. We consider two cases, n even and n odd. If n is odd, that is $n = 2k+1$ then for each i, $1 \leq i \leq n$ the values of $d(v_i/v_1)$ and

i	3	4	5	- - -	$k-1$	k	$k+1$	$k+2$	$k+3$	- - -	$n-1$	n
$d(v_i/v_1)$	2	3	4	- - -	$k-2$	$\frac{3k-2}{3}$	$k-2$	$k-3$	$k-4$	- - -	2	1
$d(v_i/v_2)$	1	2	3	- - -	$k-3$	$k-2$	$\frac{3k-2}{3}$	$k-2$	$k-3$	- - -	3	2

TABLE 2. Values of $d(v_i/v_1)$ and $d(v_i/v_2)$ when $n = 2k$

$d(v_i/v_2)$ are shown in Table 1 and if n is even, that is $n = 2k$ then the values of $d(v_i/v_1)$ and $d(v_i/v_2)$ are shown in Table 2. From the Table 1 and Table 2, it is clear that S is a rational resolving set of C_n and hence $rmd(C_n) \leq 2$. Hence the theorem.

Theorem 3.3. For any positive integer $n \geq 3$, $rmd(K_{1,n}) = n-1$

Proof: Consider the complete bipartite graph $K_{1,n}$. Let $V = \{v, v_1, v_2 \ldots v_n\}$ be the vertex set of $K_{1,n}$ such that $deg(v) = n$ and $deg(v_i) = 1$ for each i, $1 \leq i \leq n$. We observe $d(v_i/v) = \frac{1}{2}$ for each i, $1 \leq i \leq n$. Thus $v \in S$ cannot resolve any pair of vertices in $V - S$. Thus if S is a minimal resolving set then $v \notin S$.

For any $j \neq i$, $1 \leq i, j \leq n$, $d(v_j/v_i) = \frac{3}{2}$ and $d(v_i/v) = \frac{2n-1}{n+1}$.Thus $v_i \in S$ cannot resolve any pair of vertices v_l, v_m in $V - S$. Therefore if S is rational resolving set, then $|S| \geq n-1$ and hence $rmd(K_{1,n}) \geq n-1$. Let $S = \{v_1, v_2, \ldots v_{n-1}\}$, then S is a rational resolving set with $|S| = n-1$. In general, we have the following;

Theorem 3.4. For positive integers $n_1, n_2, \ldots n_k$,

$$rmd(K_{n_1,n_2,\ldots,n_k}) = n_1 + n_2 + \ldots + n_k - k.$$

4. Rational Metric Dimension of Wheels

In this section we determine rational metric dimension of wheels. The graph $W_{1,n}$ is defined for any integer $n \geq 3$, which is a graph of order $n+1$, with one vertex of degree n, which is a central vertex and all remaining vertices are of degree 3, are called rim vertices. Throughout this section we consider $V = \{v, v_1, v_2, \ldots v_n\}$ as a vertex set of $W_{1,n}$, with v as the central vertex and $v_1, v_2, \ldots v_n$ as rim vertices such that v_i is adjacent to v_{i+1}, $1 \leq i < n$.

For $n = 3$ the graph $W_{1,3}$ is a complete graph of order 4 and therefore $rmd(W_{1,3}) = 3$.

Theorem 4.1. For any integer $4 \le n \le 9$, $rmd(W_{1,n}) = 2$.

Proof: Let $S \subset V$ with $|S| = 1$. If $S = \{v\}$, then for any two vertices $v_i, v_j \in V - S$, $d(v_i/v) = d(v_j/v) = \frac{3}{4}$, therefore S is not a rational resolving set. If $S = \{v_i\}$ for some i, $1 \le i \le n$, then we can find two vertices, v_j, v_k such that $d(v_i, v_j) = d(v_i, v_k) = 2$ and $d(v_j/v_i) = d(v_j/v_k) = \frac{5}{4}$. Thus if $|S| = 1$, then S is not a resolving set. Therefore $rmd(W_{1,n}) \ge 2$. For $n = 4, 5$, $S = \{v_1, v_4\} \subset V$ is a rational resolving set of $W_{1,n}$, for $n = 6, 7, 8$ the set $S = \{v_1, v_6\}$ is a resolving set of $W_{1,n}$ and for $n = 9$, $S = \{v_1, v_7\}$ is a resolving set of $W_{1,n}$. Thus for $4 \le n \le 9$, $rmd(W_{1,n}) = 2$.

In the proof of the above theorem we observe that, $d(v_i/v) = d(v_j/v) = \frac{3}{4}$, for every $1 \le i, j \le n$ and hence the following remark

Remark 4.2. If S is a minimal resolving set of $W_{1,n}$, then $v \notin S$.

For $n \ge 10$, we determine $rmd(W_{1,n})$ with the help of the following lemma.

Lemma 4.3 For any integer $n \ge 10$, $S \subset V$ is not a rational resolving set if one of the following holds.

(1) $v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5} \notin S$, for some i.
(2) $v_i \in S$ and $v_{i-3}, v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}, v_{i+3} \notin S$, for some i.
(3) $v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4} \notin S$ and $v_j, v_{j+1}, v_{j+2}, v_{j+3}, v_{j+4} \notin S$ for some $i \ne j$.

Proof: Let S be the subset of V such that v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, $v_{i+5} \notin S$, for some i. Then for any vertex $w \in S$, $d(v_{i+1}, w) = d(v_{i+2}, w) = d(v_{i+3}, w) = d(v_{i+4}, w) = 2$ and hence $d(v_{i+2}/w) = d(v_{i+3}/w) = \frac{7}{4}$. Thus S is not a rational resolving set of G. Further if S is a subset of V such that v_{i-3}, v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}, $v_{i+3} \notin S$, for some i, then for any vertex $w \in S - \{v_i\}$, $d(v_{i-1}/w) = d(v_{i+1}/w) = \frac{7}{4}$ and $d(v_{i-1}/v_i) = d(v_{i+1}/v_i) = 1$. Hence S is not a resolving set. If there exist i, j, with v_i, v_{i+1}, v_{i+2}, v_{i+3}, $v_{i+4} \notin S$ and v_j, v_{j+1}, v_{j+2}, v_{j+3}, $v_{j+4} \notin S$, then $d(v_{i+2}/w) = d(v_{j+2}/w) = \frac{7}{4}$ for any $w \in S$. Hence S is not a resolving set in this case.

In view of the above lemma, we have the following remarks;

Remark 4.4. Let S be a resolving set of $W_{1,n}$. Suppose v_i, v_{i+1}, v_{i+2}, v_{i+3}, $v_{i+4} \notin S$ for some i, $1 \le i \le n$, then $v_{i-1}, v_{i+5} \in S$, one of $v_{i+6}, v_{i+7}, v_{i+8}$ and one of $v_{i-2}, v_{i-3}, v_{i-4}$ must belongs to S.

Remark 4.5. In any sequence of 8 consecutive rim vertices, there should be at least 2 vertices belonging to a resolving set.

Remark 4.6. While selecting vertices belonging to minimal resolving set S from V we find, between two consecutive vertices of S there will be alternately two and four vertices belonging to $V-S$, except may be in one case where there may be up to five vertices belonging to $V-S$

Theorem 4.7. For any integer $n \geq 10$,

$$rmd(W_{1,n}) = \begin{cases} \lceil \frac{n}{4} \rceil - 1 & \text{if } n \equiv 1 \pmod 8 \\ \lceil \frac{n}{4} \rceil & \text{otherwise} \end{cases}$$

.

Proof: Let S be a resolving set of $W_{1,n}$.

Claim: $|S| \geq \begin{cases} \lceil \frac{n}{4} \rceil - 1 & \text{if } n \equiv 1 \pmod 8 \\ \lceil \frac{n}{4} \rceil & \text{otherwise} \end{cases}$

In the case when $n = 8k$ or $n = 8k+1$ we have to show that 'if S is a resolving set, then $|S| \geq 2k$'. If S' is any subset of vertex set V of $W_{1,n}$ such that $|S'| \leq 2k-1$, then by the pigeon hole principle there exists at least one i, such that $|\{v_i, v_{i+1}, \dots v_{i+7}\} \cap S|$ is at most 1. By the Remark 4.5, S' is not a resolving set of $W_{1,n}$. In the case when $n = 8k+2$ or $n = 8k+3$ or $n = 8k+4$ we have to show that ' if S is a resolving set, then $|S| \geq 2k+1$'. Let S' be any subset of vertex set V of $W_{1,n}$ such that $|S'| \leq 2k$, then if S' is chosen such that it satisfies the requirements of Remarks 4.4 and Remark 4.5, then it does not satisfy the requirements of the Remark 4.6. Therefore S' is not a resolving set of $W_{1,n}$ so $|S| \geq 2k+1$. In the case when $n = 8k+5$ or $n = 8k+6$ or $n = 8k+7$ we have to show that ' if S is a resolving set, then $|S| \geq 2k+2$'. Let S' be any subset of vertex set V of $W_{1,n}$ such that $|S'| \leq 2k+1$, then by the pigeon hole principle, S' satisfies one of the conditions of the Lemma 4.3. Therefore S' is not a resolving set of $W_{1,n}$ so $|S| \geq 2k+2$. Hence the claim.

Now consider $S = \{v_1, v_4, v_9, v_{12}, \dots, v_k\} \subset V$, where

$$k = \begin{cases} n-4 & \text{if } n \equiv 0 \pmod 4 \\ n-5 & \text{if } n \equiv 1 \pmod 4 \\ n-2 & \text{if } n \equiv 2 \pmod 4 \\ n-3 & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

We prove S is a rational resolving set of $W_{1,n}$. Let $v_i, v_j \in V - S$, for some i, j with $1 < i < j < k$. Then in the view of the Lemma 4.3, $\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\} \cap S \neq \phi$. If $v_{i-2} \in S$, then $d(v_i/v_{i-2}) = \frac{3}{2}$ and $d(v_j/v_{i-2}) = \frac{7}{4}$, hence in this case v_{i-2} resolves the pair of vertices $v_i, v_j \in V - S$. If $v_{i-1} \in S$, then $d(v_i/v_{i-1}) = 1$ and

$$d(v_j/v_{i-2}) = \begin{cases} \frac{3}{2} & \text{if } j = i+1 \\ \frac{7}{4} & \text{otherwise} \end{cases}$$

Hence in this case v_{i-1} resolves the pair of vertices $v_i, v_j \in V - S$.

If $\{v_{i-2}, v_{i-1}\} \cap S = \phi$, then $\{v_{i+1}, v_{i+2}\} \cap S \neq \phi$. If $v_{i+1} \in S$ and $j = i+2$, then by the choice of S, $v_{j+1} \in S$ and $d(v_i/v_{j+1}) = \frac{3}{2}$ and $d(v_j/v_{j+1}) = 1$, hence in this case $v_{j+1} \in S$ resolves the pair of vertices $v_i, v_j \in V - S$. If $j > i+2$, then $d(v_i/v_{i+1}) = 1$, but $d(v_j/v_{i-2}) = \frac{3}{2}$ or $\frac{7}{4}$, hence in this case v_{i+1} resolves the pair of vertices $v_i, v_j \in V - S$. If $v_{i+1} \notin S$ then $v_{i+2} \in S$ and if $v_j = v_{i+1}$ or v_{i+3}, then $d(v_i/v_{i+2}) = \frac{3}{2}$ and $d(v_j/v_{i+2}) = 1$, hence in this case v_{i+2} resolves the pair of vertices $v_i, v_j \in V - S$. If $v_{i+2} \in S$ and $j > i+3$, then by the Remark 4.6 and by the choice of S one of v_{j-2} or v_{j-1} or v_{j+1} or v_{j+2} must belongs to S and that vertex resolves the pair of vertices $v_i, v_j \in V - S$.

If $k < i < j \leq n$, then if $|k-i| \leq 2$ then v_k resolves v_i, v_j and if $|k-i| > 2$, then v_1 resolves v_i, v_j. If $i < k < j$ and if $n \equiv 2 \pmod 4$ then $d(v_i/v_1) = \frac{7}{4}$ and $d(v_j/v_1) = 1$ or $\frac{3}{2}$, hence v_1 resolves v_i, v_j. Otherwise (if $n \equiv 0, 1, 3 \pmod 4$) there exist $v_l \in S$ such that $d(v_i/v_l) = 1$ or $\frac{3}{2}$ and $d(v_j/v_l) = \frac{7}{4}$, hence v_l resolves $v_i, v_j \in V - S$. Hence S is a resolving set. By the choice of S, $|S| = \lceil \frac{n}{4} \rceil - 1$ if $n \equiv 1 \pmod 8$ and $|S| = \lceil \frac{n}{4} \rceil$ otherwise. Hence the proof.

5. Rational Metric Dimension of power graphs

In this section we determine rational metric dimensions of powers of cycles and paths. The k^{th} power of a graph G, denoted by G^k, is defined on the vertices of G, with the property that two vertices in G^k are adjacent whenever $d_G(u,v) \leq k$. If $k = diam(G)$, then G^k is a complete graph and hence for a graph G of order n and diameter d, $rmd(G^d) = n-1$. For a complete graph K_n, $K_n^k \cong K_n$ and hence $rmd(K_n^k) = n-1$.

First we determine rational metric dimensions of powers of cycles. For $3 \leq n \leq 5$, the graph C_n^2 is complete and $rmd(C_n^2) = n-1$.

Theorem 5.1. For any integer $n \geq 6$,

$$rmd(C_n^2) = \begin{cases} 3 & \text{for } n = 6 \\ 2 & \text{for } n \geq 7 \end{cases}.$$

Proof: Let $G = C_n^2$ and $v_1, v_2, \ldots v_n$ be the vertices of C_n such that v_i is adjacent to v_{i+1} for $1 \leq i \leq n$. Let S be a resolving set of G. If $|S| = 1$, then $S = \{v_i\}$ for some i, $1 \leq i \leq n$ and $d(v_{i-1}/v_i) = d(v_{i+1}/v_i) = 1$, which is a contradiction to the fact that S is a resolving set. Therefore $|S| \geq 2$.

For $n = 6$, if S is resolving set, then we show that $|S| \geq 3$, i.e. we show that any set with $|S| = 2$ is not a resolving set. In C_6^2 we can choose any vertex as v_1 and remaining vertices can be chosen respectively and $d(v_1, v_j) = d(v_1, v_{6-j+2})$. Therefore it is enough to consider the cases $S = \{v_1, v_2\}$, $S = \{v_1, v_3\}$ and $S = \{v_1, v_4\}$.

Let $S = \{v_1, v_2\}$, then $d(v_6/v_1) = d(v_3/v_1) = 1$ and $d(v_6/v_2) = d(v_3/v_2) = 1$, therefore S is not a resolving set.

If $S = \{v_1, v_3\}$, then $d(v_5/v_1) = d(v_6/v_1) = 1$ and $d(v_5/v_3) = d(v_6/v_3) = 1$, therefore S is not a resolving set.

If $S = \{v_1, v_4\}$, then $d(v_2/v_1) = d(v_6/v_1) = 1$ and $d(v_2/v_4) = d(v_6/v_4) = 1$, therefore S is not a resolving set. Thus for $n = 6$, S is a resolving set only if $|S| \geq 3$. Now consider $S = \{v_1, v_2, v_3\}$. Then $V - S = \{v_4, v_5, v_6\}$.

We observe;

(1) $d(v_4/v_1) = \frac{6}{5}$ and $d(v_5/v_1) = 1$, therefore v_1 resolves v_4 and v_5.
(2) $d(v_4/v_3) = 1$ and $d(v_6/v_3) = \frac{6}{5}$, therefore v_3 resolves v_4 and v_6.
(3) $d(v_5/v_2) = \frac{6}{5}$ and $d(v_6/v_2) = 1$, therefore v_3 resolves v_4 and v_6.

Thus S is a resolving set of C_6^2 and hence $rmd(C_6^2) = 3$.

For $n \geq 7$, it is enough to construct a resolving set S with $|S| = 2$. We prove that $S = \{v_1, v_2\}$ is a resolving set. In C_n^2, we observe;

$$d(v_1, v_i) = \begin{cases} \left\lfloor \frac{i}{2} \right\rfloor & \text{for } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ \left\lfloor \frac{n-i+2}{2} \right\rfloor & \text{for } i > \left\lfloor \frac{n}{2} \right\rfloor + 1 \end{cases}$$

and

$$d(v_2,v_i)=\begin{cases}\left\lfloor\frac{i-1}{2}\right\rfloor & \text{for } 3\le i\le \left\lfloor\frac{n}{2}\right\rfloor+2\\ \left\lfloor\frac{n-i+3}{2}\right\rfloor & \text{for } i>\left\lfloor\frac{n}{2}\right\rfloor+2\end{cases}$$

Now

$$\begin{aligned} d(v_2/v_1) &= \frac{d(v_n,v_1)+d(v_1,v_1)+d(v_2,v_1)+d(v_3,v_1)+d(v_4,v_1)}{5}\\ &= \frac{1+0+1++1+2}{5}=1\end{aligned}$$

and for any i, $3\le i\le n-2$,

$$d(v_i/v_1)=\frac{1}{5}\left[d(v_{i-2},v_1)+d(v_{i-1},v_1)+d(v_i,v_1)+d(v_{i+1},v_1)+d(v_{i+2},v_1)\right]$$

We note that, for $3\le i\le\left\lfloor\frac{n}{2}\right\rfloor$, if n odd, then

$$d(v_i/v_1)=\frac{1}{5}\left[\frac{i-3}{2}+\frac{i-1}{2}+\frac{i-1}{2}+\frac{i+1}{2}+\frac{i+1}{2}\right]=\frac{5i-3}{10}$$

and if n even, then

$$d(v_i/v_1)=\frac{1}{5}\left[\frac{i-2}{2}+\frac{i-2}{2}+\frac{i}{2}+\frac{i}{2}+\frac{i+2}{2}\right]=\frac{5i-2}{10}$$

For $i=\left\lfloor\frac{n}{2}\right\rfloor$ and if i is odd, then

$$\begin{aligned} d(v_i/v_1) &= \frac{1}{5}\left[\frac{i-3}{2}+\frac{i-1}{2}+\frac{i+1}{2}+\frac{i+1}{2}+\frac{n-(i+2)+1}{2}\right]\\ &= \frac{n-3i-2}{10}=\begin{cases}\frac{5i-5}{10} & \text{if } n \text{ is even}\\ \frac{5i-4}{10} & \text{if } n \text{ is odd}\end{cases}\end{aligned}$$

For $i=\left\lfloor\frac{n}{2}\right\rfloor$ and if i is even, then

$$\begin{aligned} d(v_i/v_1) &= \frac{1}{5}\left[\frac{i-2}{2}+\frac{i-2}{2}+\frac{i}{2}+\frac{n-i}{2}+\frac{n-i}{2}\right]\\ &= \frac{2n+i-4}{10}=\begin{cases}\frac{5i-4}{10} & \text{if } n \text{ is even}\\ \frac{5i-2}{10} & \text{if } n \text{ is odd}\end{cases}\end{aligned}$$

For $i=\left\lfloor\frac{n}{2}\right\rfloor+1$ and if i is odd, then

$$\begin{aligned} d(v_i/v_1) &= \frac{1}{5}\left[\frac{i-1}{2}+\frac{i-1}{2}+\frac{n-i+1}{2}+\frac{n-i+1}{2}+\frac{n-i+1}{2}\right]\\ &= \frac{3n-i-3}{10}=\begin{cases}\frac{5i-9}{10} & \text{if } n \text{ is even}\\ \frac{5i-6}{10} & \text{if } n \text{ is odd}\end{cases}\end{aligned}$$

For $i = \left\lfloor \frac{n}{2} \right\rfloor + 1$ and if i is even, then

$$\begin{aligned} d(v_i/v_1) &= \frac{1}{5}\left[\frac{i-2}{2}+\frac{i-2}{2}+\frac{n-i+2}{2}+\frac{n-i}{2}+\frac{n-i}{2}\right] \\ &= \frac{3n-i-2}{10} = \begin{cases} \frac{5i-8}{10} & \text{if } n \text{ is even} \\ \frac{5i-5}{10} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Thus we have $d(v_i/v_1) \neq d(v_j/v_1)$ for all i, j with $i \neq j$ and $2 \leq i, j \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$.

For $i \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$, we have $d(v_i/v_1) = d(v_{n-i+2}/v_1)$. Therefore $d(v_i/v_1) = d(v_j/v_1)$ only if $j = n-i+2$. But from the expressions for $d(v_1, v_i)$ and $d(v_2, v_i)$, we observe that, whenever $d(v_i/v_1) = d(v_j/v_1)$, then $d(v_i/v_2) \neq d(v_j/v_2)$. Thus S is a resolving set of C_n^2. Hence for $n \geq 7$, $rmd(C_n^2) = 2$.

While proving the above theorem, we come across a situation where, we need to take average of some numbers which are in a nondecreasing sequence. In order to generalize the previous theorem, we need a result to compare such increasing average values. Therefore we state and prove the following lemma.

Lemma 5.2. [Sliding Lemma] If $\{a_n\}$, nondecreasing sequence in which each value is repeated r times, then for $i < j$,

$$\sum_{k=i}^{i+r} a_k < \sum_{k=j}^{j+r} a_k$$

Proof: Let $\{a_n\} = a_1, a_2, \ldots a_r, a_{r+1}, \ldots a_{2r} \ldots$, where $a_1 = a_2 = \ldots a_r < a_{r+1} = a_{r+2} \ldots = a_{2r} < a_{2r+1} = a_{2r+2} \ldots$. We observe that $a_i \leq a_j$ and for $j > i+r$, $a_i < a_j$. Therefore for $i < j$, $a_i + a_{i+1} + \ldots a_{i+r} < a_j + a_{j+1} + \ldots a_{j+r}$.

Theorem 5.3. For any integer $n \geq 3$, $rmd(P_n^2) = 1$.

Proof: By the Remark 2.3, it is enough to find a resolving set of cardinality 1. We show that $S = \{v_1\}$ is a rational resolving set of P_n^2. In the graph P_n^2,

$$d(v_1, v_i) = \begin{cases} \frac{i}{2} & \text{if } i \text{ is even} \\ \frac{i-1}{2} & \text{if } i \text{ is odd} \end{cases}$$

Therefore for $2 \leq i < j-1 \leq n$, $d(v_1, v_i) < d(v_1, v_j)$ and for $i = j-1$, $d(v_1, v_i) \leq d(v_1, v_j)$. In view of the sliding lemma , we find $d(v_i/v_1) < d(v_j/v_1)$ for $3 \leq i < j \leq n-2$. We have

$d(v_2/v_1) = 1$, $d(v_3/v_1) = \frac{6}{5}$ and hence $d(v_i/v_1) < d(v_j/v_1)$ for $2 \le i < j \le n-2$. Also we observe that,

$$\begin{aligned} d(v_{n-2}/v_1) &= \frac{d(v_{n-4},v_1)+d(v_{n-3},v_1)+d(v_{n-2},v_1)+d(v_{n-1},v_1)}{4} \\ &= \begin{cases} \frac{n-4}{2}+\frac{n-4}{2}+\frac{n-2}{2}+\frac{n-2}{2}+\frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-4}{2}+\frac{n-3}{2}+\frac{n-3}{2}+\frac{n-1}{2}+\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases} = \frac{5n-12}{10} \end{aligned}$$

$$d(v_{n-1}/v_1) = \frac{d(v_{n-3},v_1)+d(v_{n-2},v_1)+d(v_{n-1},v_1)+d(v_n,v_1)}{4} = \frac{n-2}{2}$$

and

$$d(v_n/v_1) = \frac{d(v_{n-2},v_1)+d(v_{n-1},v_1)+d(v_n,v_1)}{3} = \begin{cases} \frac{3n-5}{6} & \text{if } n \text{ is even} \\ \frac{3n-4}{6} & \text{if } n \text{ is odd} \end{cases}$$

Since $\frac{5n-12}{10} < \frac{n-2}{2} < \frac{3n-5}{6}$, for all $n \ge 3$, we have $d(v_{n-2}/v_1) < d(v_{n-1}/v_1) < d(v_n/v_1)$. Thus one can conclude that $d(v_i/v_1) < d(v_j/v_1)$ for all i, j, $2 \le i < j \le n$. Hence S is a resolving set of P_n^2 and $rmd(P_n^2) = 1$.

Theorem 5.4. For any two positive integer n, k with $n \ge k+1$, $rmd(P_n^k) = 1$.

Proof: By the Remark 2.3, it is enough to find a resolving set of cardinality 1. We show that $S = \{v_1\}$ is a rational resolving set of P_n^k. In the graph P_n^k, $d(v_1, v_i) = \lceil \frac{i-1}{k} \rceil$ for each i, $1 \le i \le n$. Therefore for $2 \le i < j \le n$, $d(v_1, v_i) \le d(v_1, v_j)$ and $d(v_1, v_i) < d(v_1, v_j)$ for $i < j-k$. For $k+1 \le i \le n-k$ we observe that

$$d(v_i/v_1) = \frac{d(v_{i-k},v_1)+d(v_{i-k+1},v_1)+\ldots+d(v_{i+k},v_1)}{2k+1} = \sum_{j=i-k}^{i+k} \frac{d(v_j,v_1)}{2k+1}$$

In the view of the sliding Lemma , we find $d(v_i/v_1) < d(v_j/v_1)$ for $k+1 \le i < j \le n-k$. Also we observe that for $2 \le i < k$,

$$d(v_i/v_1) = \sum_{j=1}^{i+k} \frac{d(v_j,v_1)}{i+k} = \frac{k+2i-2}{i+k} \quad \text{and} \quad d(v_{i+1}/v_1) = \sum_{j=1}^{i+k+1} \frac{d(v_j,v_1)}{i+k+1} = \frac{k+2i}{i+k+1}.$$

Since $\frac{k+2i}{i+k+1} - \frac{k+2i-2}{i+k} = \frac{k+2}{(i+k+1)(i+k)} > 0$, we have $d(v_i/v_1) < d(v_{i+1}/v_1)$ for each i, $2 \le i < k$.

Also, for $i \ge n-k+1$, we observe that the values of $d(v_i/v_1)$ are averages of decreasing number of terms of nondecreasing values. Therefore $d(v_i/v_1) < d(v_{i+1}/v_1)$ for $n-k+1 \le i < n$.

Hence we conclude $d(v_i/v_1) < d(v_j/v_1)$ for every i, j, with $2 \le i < j \le n$ and S is a resolving set of P_n^k. Thus $rmd(P_n^k) = 1$.

6. A characterization of RMD

In this section we obtain certain characterization for rational metric dimension of a graph. We begin with the following lemma whose proof is a direct consequence of the definition.

Lemma 6.1. If G is a graph with $rmd(G) = 1$, then there does not exist two pairs of vertices in G with same neighbourhood.

Theorem 6.2. Let $G = (V, E)$ be a graph containing at least k mutually twin vertices which induces a complete graph K_k in G. Then $rmd(G) \ge k-1$.

Proof: Let S be a resolving set of G and $w_1, w_2, \ldots, w_k$ be mutually twin vertices. For every $v \in S - \{w_1, w_2, \ldots, w_k\}$, $d(w_i/v) = \sum_{i=1}^{k} d(v, w_i) = constant$. Therefore the set $S - \{w_1, w_2, \ldots, w_k\}$ cannot resolve any pair of vertices in $\{w_1, w_2, \ldots, w_k\}$. To resolve any pair of vertices among w_i's, we need to take one of the vertices of the pair. Therefore there must be at least $k-1$ vertices of $\{w_1, w_2, \ldots, w_k\}$ must belong to a resolving set S of G. Hence $|S| \ge k-1$.

Corollary 6.3. Let $G = (V, E)$ and $V_1, V_2, \ldots V_k$ be the disjoint subsets of V such that for any two vertices $u, v \in V_i$, $N[u] = N[v]$. Then $rmd(G) \ge |V_1| + |V_2| + \ldots |V_k| - k$.

Theorem 6.4. If H is a regular graph, then $rmd(G \odot H) \ge o(G) \cdot rmd(H)$.

Proof: Let S be a resolving set of $G \odot H$ and $H_1, H_2, \ldots H_{o(G)}$ be the copies of H in $G \odot H$. For any $x, y \in H_i$, $1 \le i \le o(G)$, $d(x, v) = d(y, v)$ for any $v \in V(G \odot H) - V(H_i)$. Therefore $d(x/v) = d(y/v)$ for all $v \in S - V(H_i)$. Hence any pair of vertices of H_i are resolved by only vertices of H_i, for each i, $1 \le i \le o(G)$. Therefore $rmd(G \odot H) \ge o(G) \cdot rmd(H)$.

7. A complete list of metric dimension and rational metric dimension of connected graphs up to order five

The list below in the following table the comparison between Metric and Rational metric dimensions of all nontrivial connected graphs of order at most five, for the purpose of future study.

The graph G of Figure	$\beta(G)$	$rmd(G)$	The graph G of Figure	$\beta(G)$	$rmd(G)$
1	1	1	16	2	2
2	1	1	17	2	2
3	2	2	18	2	1
4	2	2	19	3	2
5	2	1	20	2	2
6	1	1	21	2	1
7	2	2	22	2	1
8	2	2	23	3	2
9	3	3	24	2	2
10	1	1	25	3	2
11	2	1	26	3	3
12	2	2	27	2	1
13	2	1	28	2	2
14	2	1	29	3	3
15	2	2	30	4	4

TABLE 3. Metric Dimension and Rational metric dimension of all nontrivial connected graphs up to order Five.

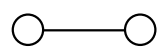


FIGURE 1

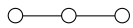


FIGURE 2

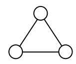


FIGURE 3

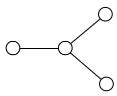


FIGURE 4

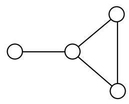


FIGURE 5

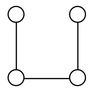


FIGURE 6

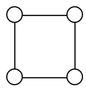


FIGURE 7

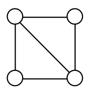
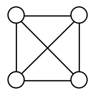


FIGURE 8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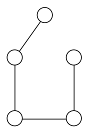
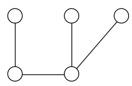


FIGURE 10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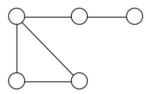
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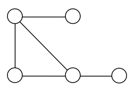
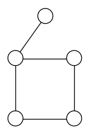


FIGURE 13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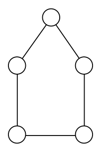
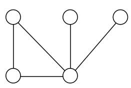


FIGURE 15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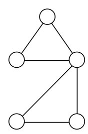
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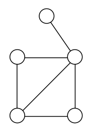
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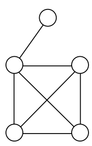
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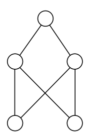
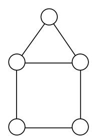


FIGURE 20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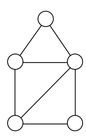
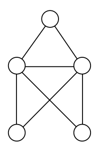


FIGURE 22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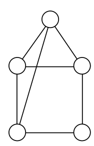
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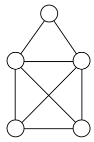

FIGURE 25

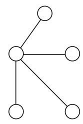

FIGURE 26

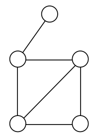

FIGURE 27

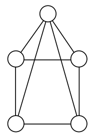

FIGURE 28

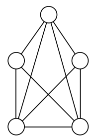

FIGURE 29

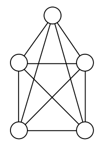

FIGURE 30

8. Open Problem

Many questions remain to be investigated, few of them are listed here.

(1) Characterize the graphs G with $rmd(G) = 1$.

(2) Characterize the graphs with $rmd(G) = \beta(G)$.

(3) For a given positive integer n, construct a graph on n vertices with $rmd(G) < \beta(G)$.

(4) Determine the integers m, n for which there is a graph G on n vertices with $rmd(G) = m$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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