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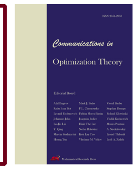

OPTIMAL CONTROL STRATEGIES FOR THE DYNAMICS OF RIFT VALLEY FEVER

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Abstract. A model to assess the impact of some control measures in the dynamics of Rift Valley Fever (RVF) is considered. We derived and analysed the conditions for optimal control of RVF with insecticides, vaccination, and personal protection using optimal control theory. We show that the control measures have a very desirable effect for reducing the number of infected individuals and that multiple controls are more effective than single control. Moreover, we show that effective and optimal use of insecticides and personal protection without the use of vaccination is not beneficial if total elimination of the disease is desirable in the community.

Keywords: Rift valley fever; optimal control; insecticide; larvicide; adulticide; vaccination; personal protection.

2010 AMS Subject Classification: 92B05, 37N205.

1. Introduction

Rift Valley Fever (RVF) is a viral disease which affects both animals and humans. The disease was first reported in Kenya (Africa) in 1931 [1]. RVF is primarily transmitted in both animals and humans by the bite of infected mosquitoes, mainly *Aedes* and *Culex* spp. which can acquire

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the virus from feeding on an infected animal [1-3]. The female *Aedes* spp. mosquito is also capable of transmitting the virus directly to her off-spring via eggs leading to new generations of infected mosquitoes hatching from eggs [4].

The epidemiological model of RVF which was first proposed in Gaff et al. [5], and later modified in Gaff et al [6] has influenced both modeling and application of control strategies in RVF. Recently, Mpeshe et al. [7] developed a model to assess the impact of climate change, mainly precipitation and temperature on the dynamics of RVF. While it is clear that the basic reproduction number $(\mathcal{R}_0)$, which is the initial transmission of the disease, increases with increase in rainfall, it is not the case for temperature where they experience high $\mathcal{R}_0$ for low temperatures.

Many studies have been carried out to combat vector-borne diseases including RVF. Most studies in RVF had focused on the role of disease transmission parameters in the reduction of $\mathcal{R}_0$ and disease prevalence [5 - 12]. These studies focus on formulation of mathematical models to predict long term behaviour and only few of them did account for control measures [6, 8]. While optimal control theory have been applied to number of studies in mathematical models of vector-borne diseases including malaria and chikungunya [13-16] hardly have been done for RVF. The optimal control model presented by Adongo et al. [17] dealt with quantitative and qualitative vaccination control strategies to effectively reduce the transmission of the disease among livestock with minimization of the infected mosquitoes. Though the model considered only one strategy, it saves a first time model on optimal control strategies for RVF.

In this article, we formulate an optimal control model for RVF in order to derive optimal control strategies with minimal implementation cost. The mathematical foundation of the control model and the derivation of the optimal control variables are given using Pontryagin's Maximum Principle [18] and the proof for the existence of the optimal control is established using Fleming and Rishel [19] and Lukes [20].

2. Materials and Methods

2.1 Formulation of the optimal control model

To construct a deterministic model for RVF transmission with control term, the model of Mpeshe et al. [7] was modified and optimal control terms were added as shown in Figure 1. The

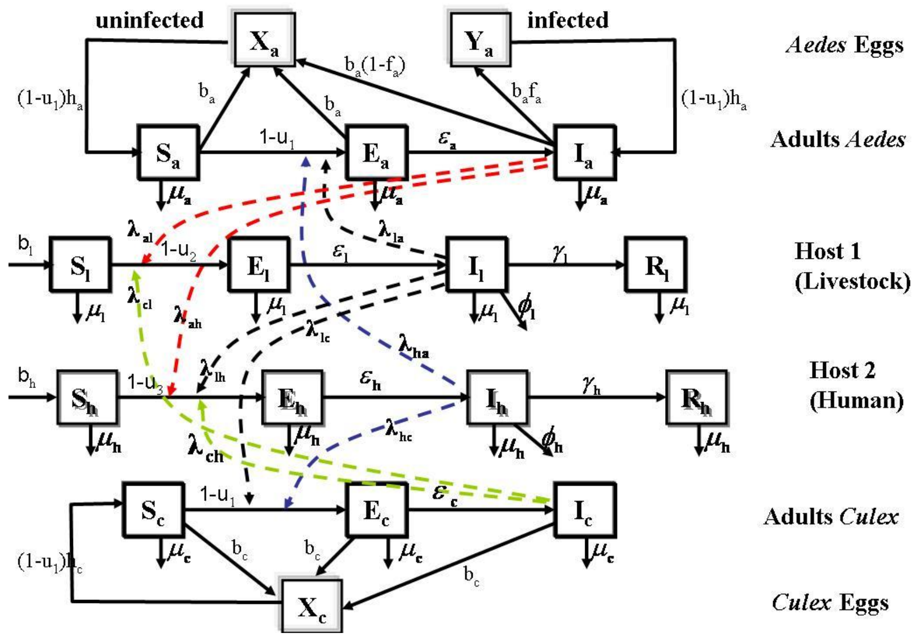


FIGURE 1. Flow diagram for the optimal control model

model considers three populations: mosquitoes, livestock, and humans with disease-dependent death rate for livestock and humans. The mosquito population is subdivided into two: *Aedes* species and *Culex* species. Due to vertical transmission in *Aedes* spp., the model include both infected and uninfected eggs. The egg population of *Aedes* spp. consists of uninfected eggs (X_a) and infected eggs (Y_a). The population for adult *Aedes* spp. consists of susceptible adults (S_a), latently infected adults (E_a), and infectious adults (I_a). The egg population of *Culex* spp. consists of uninfected eggs (X_c) only and the population for adult *Culex* spp. consists of susceptible adults (S_c), latently infected adults (E_c), and infectious adults (I_c). The livestock population consists of susceptible livestock (S_l), latently infected livestock (E_l), infectious livestock (I_l), and recovered livestock (R_l). The human population consists of susceptible humans (S_h), latently infected humans (E_h), infectious humans (I_h) and recovered humans (R_h). T and P represent temperature and precipitation respectively. The model parameters and their description as they have been used in this work are given in Table 1.

The optimal control model considers three controls: insecticides which include larvicide and adulticide in mosquitoes, vaccination in livestock, and prevention or personal protection in humans. We employ the control mechanisms $u_i(t)$ in all populations, where $1 - u_i(t)$ is the failure rate of the control mechanism $u_i(t)$ for $i = 1, 2, 3$. The control mechanism $u_1(t)$ for mosquitoes represents the use of insecticides, mainly larvicide and adulticde, the control mechanism $u_2(t)$ for animals represent the effort of vaccination, and the control mechanism $u_3(t)$ in humans represent the effort of prevention or personal protection. It is assumed that the application of insecticides will also increase the death rate of mosquitoes in each compartment at a rate proportional to $u_1(t)$. We take these rates to be $\mu_a(T)u_1(t)$ for *Aedes* and $\mu_c(T)u_1(t)$ for *Culex* so that when the control is 100% effective, the death rate in mosquitoes is doubled. Thus, we construct the optimal model as follows:

Aedes Mosquito

$$\frac{dX_a}{dt} = b_a(T,P)(N_a - f_aI_a) - (1-u_1(t))h_a(T,P)X_a, \tag{1a}$$

$$\frac{dY_a}{dt} = b_a(T,P)f_aI_a - (1-u_1(t))h_a(T,P)Y_a, \tag{1b}$$

$$\frac{dS_a}{dt} = (1-u_1(t))h_a(T,P)X_a - \mu_a(T)(1+u_1(t))S_a - (1-u_1(t))(\lambda_{la}(T)\frac{I_l}{N_l}S_a + \lambda_{ha}(T)\frac{I_h}{N_h}S_a), \tag{1c}$$

$$\frac{dE_a}{dt} = (1-u_1(t))(\lambda_{la}(T)\frac{I_l}{N_l}S_a + \lambda_{ha}(T)\frac{I_h}{N_h}S_a) - (\varepsilon_a(T) + \mu_a(T)(1+u_1(t)))E_a, \tag{1d}$$

$$\frac{dI_a}{dt} = (1-u_1(t))h_a(T,P)Y_a + \varepsilon_a(T)E_a - \mu_a(T)(1+u_2(t))I_a. \tag{1e}$$

Culex Mosquito

$$\frac{dX_c}{dt} = b_c(T,P)N_c - (1-u_1(t))h_c(T,P)X_c, \tag{2a}$$

$$\frac{dS_c}{dt} = (1-u_1(t))h_c(T,P)X_c - \mu_c(T)(1+u_1(t))S_c - (1-u_1(t))(\lambda_{lc}(T)\frac{I_l}{N_l}S_c + \lambda_{hc}(T)\frac{I_h}{N_h}S_c), \tag{2b}$$

$$\frac{dE_c}{dt} = (1-u_1(t))(\lambda_{lc}(T)\frac{I_l}{N_l}S_c + \lambda_{hc}(T)\frac{I_h}{N_h}S_c) - (\varepsilon_c(T) + \mu_c(T)(1+u_1(t)))E_c, \tag{2c}$$

$$\frac{dI_c}{dt} = \varepsilon_c(T)E_c - \mu_c(T)(1+u_2(t))I_c. \tag{2d}$$

Livestock

$$\frac{dS_l}{dt} = b_lN_l - \mu_lS_l - (1-u_2(t))(\lambda_{al}(T)\frac{I_a}{N_a}S_l + \lambda_{cl}(T)\frac{I_c}{N_c}S_l), \tag{3a}$$

$$\frac{dE_l}{dt} = (1-u_2(t))(\lambda_{al}(T)\frac{I_a}{N_a}S_l + \lambda_{cl}(T)\frac{I_c}{N_c}S_l) - (\varepsilon_l + \mu_l)E_l, \tag{3b}$$

$$\frac{dI_l}{dt} = \varepsilon_lE_l - (\mu_l + \phi_l + \gamma_l)I_l, \tag{3c}$$

$$\frac{dR_l}{dt} = \gamma_lI_l - \mu_lR_l. \tag{3d}$$

Humans

$$\frac{dS_h}{dt} = b_h N_l - \mu_h S_h - (1-u_3(t))(\lambda_{lh}\frac{I_l}{N_l}S_h + \lambda_{ah}(T)\frac{I_a}{N_a}S_h + \lambda_{ch}(T)\frac{I_c}{N_c}S_h), \tag{4a}$$

$$\frac{dE_h}{dt} = (1-u_3(t))(\lambda_{lh}\frac{I_l}{N_l}S_h + \lambda_{ah}(T)\frac{I_a}{N_a}S_h + \lambda_{ch}(T)\frac{I_c}{N_c}S_h) - (\varepsilon_h + \mu_h)E_h, \tag{4b}$$

$$\frac{dI_h}{dt} = \varepsilon_h E_h - (\mu_h + \phi_h + \gamma_h)I_h, \tag{4c}$$

$$\frac{dR_h}{dt} = \gamma_h I_h - \mu_h R_h. \tag{4d}$$

2.2 The optimal control problem

Our aim is to minimise the number of actively infected livestock and humans while minimising the the cost of control mechanism $u_i(t)$. Therefore for a fixed terminal time t_f, the problem is to minimise the cost objective functional

$$J(u_1,u_2,u_3) = \int_0^{t_f} (A_1 I_l + A_2 I_h + B_1 u_1^2 + B_2 u_2^2 + B_3 u_3^2)dt, \tag{5}$$

where t_f is the final time, A_1 and A_2 are positive weight constants of the actively infected livestock and humans, and B_1, B_2, and B_3 positive weight constants for the control mechanism which regularise the optimal control.

We assume that the cost of each control mechanism is quadratic in the cost function, that is, $B_1 u_1^2$ is the cost of the control mechanism in mosquitoes which could be associated with the application of insecticides, $B_2 u_2^2$ is the cost of the control mechanism in livestock mainly associated with vaccination, and $B_3 u_3^2$ is the cost of the control mechanism in humans associated with prevention measures or personal protection such as educating the public about person protection, hygiene, the basic needs like gloves for people handling animals, use of mosquito treated bed nets, and avoiding contact with infected livestock.

Therefore, we seek to obtain an optimal control u_1^*, u_2^*, u_3^* such that

$$J(u_1^*, u_2^*, u_3^*) = \min_{\Omega} J(u_1,u_2,u_3), \tag{6}$$

where $\Omega = \{(u_1,u_2,u_3) \in L^1(0,t_f) | a_i \leq u_i \leq b_i,\ i = 1,2,3\, t \in [0,\ t_f]\}$, and a_i, b_i are fixed positive constants.

As a basic framework, we will need to state and prove the existence of the optimal control and then characterise the optimal control through the optimality system.

2.3 Existence of an optimal control

Here we state and prove the existence of optimal control using the existence results from Fleming and Rishel [19], and Lukes [20].

Theorem 2.1. *Consider the optimal control problem with state equations. There exists* $(u_1^*, u_2^*, u_3^*) \in \Omega$ *such that*

$$\min_{\Omega} J(u_1, u_2, u_3) = J(u_1^*, u_2^*, u_3^*). \tag{7}$$

Proof. To use the existence results from Fleming and Rishel [19] [Theorem 4.1. pg 68 - 70] we first need to check the following properties:

(1) The set of controls and the corresponding state variables is non-empty.
(2) The control set Ω is convex and closed.
(3) The right hand side of the state system is bounded by a linear function in the state control.
(4) The integrand of the objective functional is convex.
(5) There exists $c_1, c_2 > 0$ and $\beta > 1$ such that the integrand of the objective functional is bounded below by $c_1(|u_1|^2 + |u_2|^2 + |u_3|^2)^{\beta/2} - c_2$.

The existence results in Lukes [20] [Theorem 9.2.1 pg 182] for the state system verify that the first property is satisfied. By definition of convex set, the control set Ω is convex and closed, hence, the second property is also satisfied. Since the state solutions of a linear state system in u_i are bounded, then, the right hand side is bounded by a linear function. Finally, there are $c_1, c_2 > 0$ and $\beta > 1$ satisfying $A_1 I_l + A_2 I_h + B_1 u_1^2(t) + B_2 u_2^2(t) + B_3 u_3^2(t) \geq c_1(|u_1|^2 + |u_2|^2 + |u_3|^2)^{\beta/2} - c_2$ because the state variables are bounded. Hence, the existence of optimal control follows from the existence results by Fleming and Rishel [19]. □

2.4 Characterization of an optimal control

The existence of the optimal control leads to the derivation of the optimality system. Using the Pontryagin's Maximum Principle [18] we define the Hamiltonian H below and derive the necessary conditions. That is,

$$H = A_1 I_l + A_2 I_h + B_1 u_1^2(t) + B_2 u_2^2(t) + B_3 u_3^2(t) + \sum_P \lambda_P f_P, \tag{8}$$

where P is the set of state variables, λ_P is the adjoint variable of the P_{th} state variable, and f_P is the right hand side of the differential equation of the P_{th} state variable.

Let P be the set of state variables, Ω be set of controls, and Q be the set of adjoint variables. The Lagrangian function for our problem consists of the integrand of the objective functional, and the inner product of the right hand side of the state equations and the adjoint variables $(\lambda_{X_a}, \lambda_{Y_a}, \cdots, \lambda_{R_h})$. In more compact form we define the Lagrangian by

$$\mathcal{L} = H - \sum_{i=1}^{3} w_{ij}(u_i(t) - a_i) - \sum_{i=1}^{3} w_{ij}(b_i - u_i(t)), \quad \text{for} \quad j = 1, 2 \tag{9}$$

where $w_{ij}(t) \geq 0$ are penalty multipliers satisfying

$$w_{11}(u_1(t) - a_1) = w_{12}(b_1 - u_1(t)) = 0 \quad \text{at} \quad \text{optimal} \quad \text{control} \quad u_1^*, \tag{10a}$$

$$w_{21}(u_2(t) - a_2) = w_{22}(b_2 - u_2(t)) = 0 \quad \text{at} \quad \text{optimal} \quad \text{control} \quad u_2^*, \tag{10b}$$

$$w_{31}(u_3(t) - a_3) = w_{32}(b_3 - u_3(t)) = 0 \quad \text{at} \quad \text{optimal} \quad \text{control} \quad u_3^*. \tag{10c}$$

The expanded form of the Lagrangian is given by

$$\begin{aligned} \mathcal{L} = {} & A_1 I_l + A_2 I_h + B_1 u_1^2(t) + B_2 u_2^2(t) + B_3 u_3^2(t) \\ & + \lambda_{X_a}[b_a(T,P)(N_a - f_a I_a) - (1 - u_1(t))h_a(T,P)X_a] \\ & + \lambda_{Y_a}[b_a(T,P) f_a I_a - (1 - u_1(t))h_a(T,P)Y_a] \end{aligned} \tag{11}$$

$$+\lambda_{S_a}[(1-u_1(t))h_a(T,P)X_a-\mu_a(T)(1+u_1(t))S_a-(1-u_1(t))(\lambda_{la}(T)\frac{I_l}{N_l}+\lambda_{ha}(T)\frac{I_h}{N_h})S_a]$$

$$+\lambda_{E_a}[(1-u_1(t))(\lambda_{la}(T)\frac{I_l}{N_l}+\lambda_{ha}(T)\frac{I_h}{N_h})S_a-(\varepsilon_a(T)+\mu_a(T)(1+u_1(t)))E_a]$$

$$+\lambda_{I_a}[(1-u_1(t))h_a(T,P)Y_a+\varepsilon_a(T)E_a-\mu_a(T)(1+u_1(t))I_a]$$

$$+\lambda_{X_c}[b_c(T,P)N_c-(1-u_1(t))h_c(T,P)X_c]$$

$$+\lambda_{S_c}[(1-u_1(t))h_c(T,P)X_c-\mu_c(T)(1+u_1(t))S_c-(1-u_1(t))(\lambda_{lc}(T)\frac{I_l}{N_l}+\lambda_{hc}(T)\frac{I_h}{N_h})S_c]$$

$$+\lambda_{E_c}[(1-u_1(t))(\lambda_{lc}(T)\frac{I_l}{N_l}+\lambda_{hc}(T)\frac{I_h}{N_h})S_c-(\varepsilon_c(T)+\mu_c(T)(1+u_1(t)))E_c]$$

$$+\lambda_{I_c}[\varepsilon_c(T)E_c-\mu_c(T)(1+u_1(t))I_c]$$

$$+\lambda_{S_l}[b_lN_l-\mu_lS_l-(1-u_2(t))(\lambda_{al}(T)\frac{I_a}{N_a}+\lambda_{cl}(T)\frac{I_c}{N_c})S_l]$$

$$+\lambda_{E_l}[(1-u_2(t))(\lambda_{al}(T)\frac{I_a}{N_a}+\lambda_{cl}(T)\frac{I_c}{N_c})S_l-(\varepsilon_l+\mu_l)E_l]$$

$$+\lambda_{I_l}[\varepsilon_lE_l-(\mu_l+\phi_l+\gamma_l)I_l]+\lambda_{R_l}[\gamma_lI_l-\mu_lR_l]$$

$$+\lambda_{S_h}[b_hN_h-\mu_hS_h-(1-u_3(t))(\lambda_{lh}\frac{I_l}{N_l}+\lambda_{ah}(T)\frac{I_a}{N_a}+\lambda_{ch}(T)\frac{I_c}{N_c})S_h]$$

$$+\lambda_{E_h}[(1-u_3(t))(\lambda_{lh}\frac{I_l}{N_l}+\lambda_{ah}(T)\frac{I_a}{N_a}+\lambda_{ch}(T)\frac{I_c}{N_c})S_h-(\varepsilon_h+\mu_h)E_h]$$

$$+\lambda_{I_h}[\varepsilon_hE_h-(\mu_h+\phi_h+\gamma_h)I_h]+\lambda_{R_h}[\gamma_hI_h-\mu_hR_h]-w_{11}(u_1(t)-a_1)-w_{12}(b_1-u_1(t))$$

$$-w_{21}(u_2(t)-a_2)-w_{22}(b_2-u_2(t))-w_{31}(u_3(t)-a_3)-w_{32}(b_3-u_3(t)).$$

Theorem 2.2. *Given u_i^* the set optimal control, and P^* the corresponding set of solution of the state system that minimises J over Ω, then there exists adjoint variables λ_P such that*

(12a) $$\frac{d\lambda_P}{dt}=-\frac{\partial L}{\partial P}\quad \textit{adjoint}\quad \textit{conditions},\quad \textit{and}$$

(12b) $$\lambda_P(t_f)=0\quad \textit{transversality}\quad \textit{conditions}.\quad \textit{Furthermore},$$

(12c) $$\frac{\partial L}{\partial u_i}=0\quad \text{at}\quad u_i^*=0\quad \textit{optimality}\quad \textit{conditions}.$$

Proof. The adjoint system is obtained by taking the partial derivative of the Lagrangian $\mathcal{L}$ with respect to states variables. That is,

$$\frac{d\lambda_{X_a}}{dt} = (\lambda_{X_a} - \lambda_{S_a})(1-u_1(t))h_a(T,P), \tag{13a}$$

$$\frac{d\lambda_{Y_a}}{dt} = (\lambda_{Y_a} - \lambda_{I_a})(1-u_1(t))h_a(T,P), \tag{13b}$$

$$\frac{d\lambda_{S_a}}{dt} = \lambda_{S_a}\mu_a(T)(1+u_1(t)) + (\lambda_{S_a} - \lambda_{E_a})(1-u_1(t))(\lambda_{la}(T)\frac{I_l}{N_l} + \lambda_{ha}(T)\frac{I_h}{N_h}), \tag{13c}$$

$$\frac{d\lambda_{E_a}}{dt} = \lambda_{E_a}\mu_a(T)(1+u_1(t)) + (\lambda_{E_a} - \lambda_{I_a})\varepsilon_a(T), \tag{13d}$$

$$\begin{aligned}\frac{d\lambda_{I_a}}{dt} &= (\lambda_{X_a} - \lambda_{Y_a})b_a(T,P)f_a + \lambda_{I_a}\mu_a(T)(1+u_1(t)) \\ &\quad + (\lambda_{S_l} - \lambda_{E_l})(1-u_2(t))\lambda_{al}(T)\frac{S_l}{N_a} + (\lambda_{S_h} - \lambda_{E_h})(1-u_3(t))\lambda_{ah}(T)\frac{S_h}{N_a},\end{aligned} \tag{13e}$$

$$\frac{d\lambda_{X_c}}{dt} = (\lambda_{X_c} - \lambda_{S_c})(1-u_1(t))h_c(T,P), \tag{13f}$$

$$\frac{d\lambda_{S_c}}{dt} = \lambda_{S_c}\mu_c(T)(1+u_1(t)) + (\lambda_{S_c} - \lambda_{E_c})(1-u_1(t))(\lambda_{lc}(T)\frac{I_l}{N_l} + \lambda_{hc}(T)\frac{I_h}{N_h}) \tag{13g}$$

$$\frac{d\lambda_{E_c}}{dt} = \lambda_{E_c}\mu_c(T)(1+u_1(t)) + (\lambda_{E_c} - \lambda_{I_c})\varepsilon_c(T), \tag{13h}$$

$$\begin{aligned}\frac{d\lambda_{I_c}}{dt} &= \lambda_{I_c}\mu_c(T)(1+u_1(t)) + (\lambda_{S_l} - \lambda_{E_l})(1-u_2(t))\lambda_{cl}(T)\frac{S_l}{N_c} \\ &\quad + (\lambda_{S_h} - \lambda_{E_h})(1-u_3(t))\lambda_{ch}(T)\frac{S_h}{N_c},\end{aligned} \tag{13i}$$

$$\frac{d\lambda_{S_l}}{dt} = \lambda_{S_l}\mu_l + (\lambda_{S_l} - \lambda_{E_l})(1-u_2(t))(\lambda_{al}(T)\frac{I_a}{N_a} + \lambda_{cl}(T)\frac{I_c}{N_c}), \tag{13j}$$

$$\frac{d\lambda_{E_l}}{dt} = \lambda_{E_l}\mu_l + (\lambda_{E_l} - \lambda_{I_l})\varepsilon_l, \tag{13k}$$

$$\begin{aligned}\frac{d\lambda_{I_l}}{dt} &= (\lambda_{S_a} - \lambda_{E_a})(1-u_1(t))\lambda_{la}(T)\frac{S_a}{N_l} + (\lambda_{S_c} - \lambda_{E_c})(1-u_1(t))\lambda_{lc}(T)\frac{S_c}{N_l}, \\ &\quad + (\lambda_{S_h} - \lambda_{E_h})(1-u_3(t))\lambda_{lh}(T)\frac{S_h}{N_l} + \lambda_{I_l}(\mu_l + \phi_l) + (\lambda_{I_l} - \lambda_{R_l})\gamma_l - A_1,\end{aligned} \tag{13l}$$

$$\frac{d\lambda_{R_l}}{dt} = \lambda_{R_l}\mu_l, \tag{13m}$$

$$\frac{d\lambda_{S_h}}{dt} = \lambda_{S_h}\mu_h + (\lambda_{S_h} - \lambda_{E_h})(1-u_3(t))(\lambda_{ah}(T)\frac{I_a}{N_a} + \lambda_{ch}(T)\frac{I_c}{N_c} + \lambda_{lh}(T)\frac{I_l}{N_l}), \tag{13n}$$

$$\frac{d\lambda_{E_h}}{dt} = \lambda_{E_h}\mu_h + (\lambda_{E_h} - \lambda_{I_h})\varepsilon_h, \tag{13o}$$

$$\begin{aligned}\frac{d\lambda_{I_h}}{dt} &= (\lambda_{S_a} - \lambda_{E_a})(1-u_1(t))\lambda_{ha}(T)\frac{S_a}{N_h} + (\lambda_{S_c} - \lambda_{E_c})(1-u_1(t))\lambda_{hc}(T)\frac{S_c}{N_h}, \\ &\quad + \lambda_{I_h}(\mu_h + \phi_h) + (\lambda_{I_h} - \lambda_{R_h})\gamma_h - A_2,\end{aligned} \tag{13p}$$

$$\frac{d\lambda_{R_h}}{dt} = \lambda_{R_h}\mu_h. \tag{13q}$$

To determine the optimal solution for our Lagrangian, we first take the partial derivative of $\mathcal{L}$ with respect to u_i for $i = 1, 2, 3$. That is,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial u_1} &= 2B_1u_1 - (\lambda_{X_a} - \lambda_{S_a})h_a(T,P)X_a + (\lambda_{Y_a} - \lambda_{I_a})h_a(T,P)Y_a \\ &\quad - \mu_a(\lambda_{S_a}S_a + \lambda_{E_a}E_a + \lambda_{I_a}I_a) + (\lambda_{S_a} - \lambda_{E_a})(\lambda_{la}(T)\frac{I_l}{N_l}S_a + \lambda_{ha}(T)\frac{I_h}{N_h}S_a) \\ &\quad + (\lambda_{X_c} - \lambda_{S_c})h_c(T,P)X_c - \mu_c(\lambda_{S_c}S_c + \lambda_{E_c}E_c + \lambda_{I_c}I_c) \\ &\quad + (\lambda_{S_c} - \lambda_{E_c})(\lambda_{lc}(T)\frac{I_l}{N_l}S_c + \lambda_{hc}(T)\frac{I_h}{N_h}S_c) - w_{11} + w_{12},\end{aligned} \tag{14a}$$

$$\frac{\partial \mathcal{L}}{\partial u_2} = 2B_2u_2 + (\lambda_{S_l} - \lambda_{E_l})(\lambda_{al}(T)\frac{I_a}{N_a}S_l + \lambda_{cl}(T)\frac{I_c}{N_c}S_l) - w_{21} + w_{22}, \tag{14b}$$

$$\frac{\partial \mathcal{L}}{\partial u_3} = 2B_3u_3 + (\lambda_{S_h} - \lambda_{E_h})(\lambda_{ah}(T)\frac{I_a}{N_a}S_h + \lambda_{ch}(T)\frac{I_c}{N_c}S_h + \lambda_{lh}\frac{I_l}{N_l}S_h) - w_{31} + w_{32}. \tag{14c}$$

Setting $\frac{\partial \mathcal{L}}{\partial u_i} = 0$ at $u_i^*(t)$, we have

$$\begin{aligned}u_1^*(t) &= \frac{1}{2B_1}[(\lambda_{S_a} - \lambda_{X_a})h_a(T,P)X_a + (\lambda_{I_a} - \lambda_{Y_a})h_a(T,P)Y_a + (\lambda_{S_c} - \lambda_{X_c})h_c(T,P)X_c \\ &\quad + \mu_a(\lambda_{S_a}S_a + \lambda_{E_a}E_a + \lambda_{I_a}I_a) + \mu_c(\lambda_{S_c}S_c + \lambda_{E_c}E_c + \lambda_{I_c}I_c) \\ &\quad + (\lambda_{E_a} - \lambda_{S_a})(\lambda_{la}(T)\frac{I_l}{N_l}S_a + \lambda_{ha}(T)\frac{I_h}{N_h}S_a) \\ &\quad + (\lambda_{E_c} - \lambda_{S_c})(\lambda_{lc}(T)\frac{I_l}{N_l}S_c + \lambda_{hc}(T)\frac{I_h}{N_h}S_c) + w_{11} - w_{12}],\end{aligned} \tag{15a}$$

$$u_2^*(t) = \frac{1}{2B_2}[(\lambda_{E_l} - \lambda_{S_l})(\lambda_{al}(T)\frac{I_a}{N_a}S_l + \lambda_{cl}(T)\frac{I_c}{N_c}S_l) + w_{21} - w_{22}], \tag{15b}$$

$$u_3^*(t) = \frac{1}{2B_3}[(\lambda_{E_h} - \lambda_{S_h})(\lambda_{ah}(T)\frac{I_a}{N_a}S_h + \lambda_{ch}(T)\frac{I_c}{N_c}S_h + \lambda_{lh}\frac{I_l}{N_l}S_h) + w_{31} - w_{32}]. \tag{15c}$$

Applying the standard optimality technique involving the bounds of control, we have u_i^* independent of w_{ij} as follows:

$$\begin{aligned}u_1^*(t) &= \min\{b_1, \max\{a_1, \frac{1}{2B_1}[(\lambda_{S_a} - \lambda_{X_a})h_a(T,P)X_a + (\lambda_{I_a} - \lambda_{Y_a})h_a(T,P)Y_a \\ &\quad + (\lambda_{S_c} - \lambda_{X_c})h_c(T,P)X_c + (\lambda_{E_a} - \lambda_{S_a})(\lambda_{la}(T)\frac{I_l}{N_l}S_a + \lambda_{ha}(T)\frac{I_h}{N_h}S_a) \\ &\quad + (\lambda_{E_c} - \lambda_{S_c})(\lambda_{lc}(T)\frac{I_l}{N_l}S_c + \lambda_{hc}(T)\frac{I_h}{N_h}S_c) \\ &\quad + \mu_a(\lambda_{S_a}S_a + \lambda_{E_a}E_a + \lambda_{I_a}I_a) + \mu_c(\lambda_{S_c}S_c + \lambda_{E_c}E_c + \lambda_{I_c}I_c)]\}\},\end{aligned} \tag{16a}$$

$$u_2^*(t) = \min\{b_2, \max\{a_2, \frac{1}{2B_2}[(\lambda_{E_l} - \lambda_{S_l})(\lambda_{al}(T)\frac{I_a}{N_a}S_l + \lambda_{cl}(T)\frac{I_c}{N_c}S_l)]\}\}, \tag{16b}$$

$$u_3^*(t) = \min\{b_3, \max\{a_4, \frac{1}{2B_3}[(\lambda_{E_h} - \lambda_{S_h})(\lambda_{ah}(T)\frac{I_a}{N_a}S_h + \lambda_{ch}(T)\frac{I_c}{N_c}S_h + \lambda_{lh}\frac{I_l}{N_l}S_h)]\}\}. \quad \square \tag{16c}$$

The optimality system consists of the state system together with the adjoint system, the initial and transversality conditions, and the optimality conditions.

3. Results and Discussion

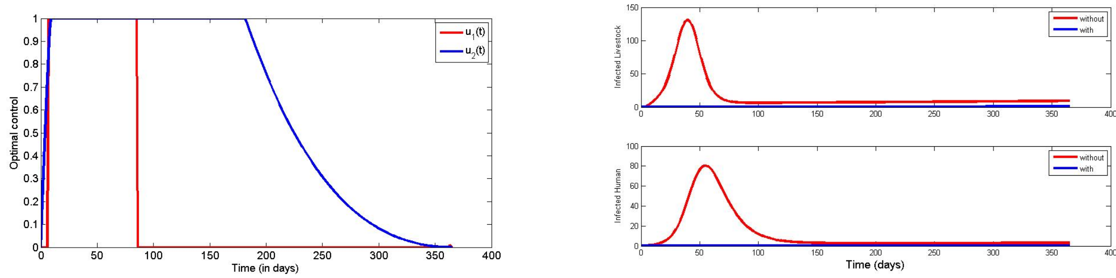
In this section, we present the results for the optimal control strategies without considering the effect of temperature and precipitation in the simulations. We solve the optimal control problem comprising of the model equations, the adjoint equations and the control mechanisms variables using fourth order Runge-Kutta scheme. Indeed the state system is solved forward in time with initial conditions $X_a(0) = 999, Y_a(0) = 1, S_a(0) = 980, E_a(0) = 20, I_a(0) = 0, X_c(0) = 1000, S_c(0) = 980, E_c(0) = 20, I_c(0) = 0, S_l(0) = 999, E_l(0) = 1, I_l(0) = 0, R_l(0) = 0, S_h(0) = 999, E_h(0) = 1, I_h(0) = 0$, and $R_h(0) = 0$, while the adjoint system is solved backward in time with terminal conditions $\lambda_P(t_f) = 0$ where $t_f = 365$ days. The controls are considered to be bounded in the interval $[0, 1]$ and the weights in the objective functional are chosen to be $A_1 = A_2 = 1000, B_1 = 0.0001, B_2 = 1000, B_3 = 0.1$. These weights are theoretically chosen just to reveal the control strategies proposed in this study because computation of real weights is very involving and needs a lot of information. The parameter values used in the simulations are given in Table 1. We use various combinations of the three controls at a time to investigate and compare their numerical results from simulations.

3.1 Optimal insecticides and livestock vaccination strategy

With this strategy, the control u_1 on mosquitoes and u_2 on livestock are both used to optimise the objective function J, while u_3 is set to zero. Figure 2(a) shows the optimal controls u_1 and u_2 when applied to optimise the objective function J. In this case, the control u_1 is at upper bound from $t = 7$ to $t = 89$ days before it drops rapidly to zero. The control u_2 is also at upper bound from $t = 9$ to $t = 181$ days before it gradually start to decrease up zero at the final time. Figure 2(b) shows the number of infected livestock and humans over time with and without control. The figures indicates that there is a significant difference in the number of infecteds with and without control. This means that an effective and optimal use of insecticides in mosquitoes and vaccination in livestock may be beneficial even without the use of personal protection.

TABLE 1. Values of parameters used in the optimal model of RVF. These values are the maximum baseline values for each parameter.

Parameter	Description	Value	Reference
$1/b_a$	number of *Aedes* eggs laid per day	200	estimate
$1/b_c$	number of *Culex* eggs laid per day	200	estimate
$1/h_a$	development time of *Aedes*	20	estimate
$1/h_c$	development time of *Culex*	20	estimate
b_h	daily birthrate in humans	0.0015	estimate
b_l	daily birthrate in livestock	0.0025	estimate
$1/\mu_a$	lifespan of *Aedes*	60 days	[5]
$1/\mu_c$	lifespan of *Culex*	60 days	[5]
$1/\mu_h$	lifespan of humans	60 yrs	[11]
$1/\mu_l$	lifespan of livestock	10 yrs	[21]
$1/\varepsilon_a$	incubation period of *Aedes*	8 days	[11]
$1/\varepsilon_c$	incubation period of *Culex*	8 days	[11]
$1/\varepsilon_h$	incubation period of humans	6 days	[11]
$1/\varepsilon_l$	incubation period of livestock	6 days	[11]
ϕ_l	disease induced death rate in livestock	0.10	[11]
ϕ_h	disease induced death rate in humans	0.10	[11]
$1/\gamma_l$	infectious period in livestock	5 days	[11]
$1/\gamma_h$	infectious period in humans	7 days	[11]
λ_{al}	adequate contact rate: *Aedes* to livestock	0.48	[6]
λ_{cl}	adequate contact rate: *Culex* to livestock	0.13	[6]
λ_{la}	adequate contact rate: livestock to *Aedes*	0.395	[6]
λ_{lc}	adequate contact rate: livestock to *Culex*	0.56	[6]
λ_{ah}	adequate contact rate: *Aedes* to humans	0.025	estimate
λ_{ch}	adequate contact rate: *Culex* to humans	0.065	estimate
λ_{ha}	adequate contact rate: humans to *Aedes*	0.0125	estimate
λ_{hc}	adequate contact rate: humans to *Culex*	0.025	estimate
λ_{lh}	adequate contact rate: livestock to humans	0.002	[11]
f_a	vertical transmission rate in *Aedes*	0.1	[6]



(a) optimal controls u_1 and u_2 (b) Infecteds with and without control

FIGURE 2. Optimal insecticides and livestock vaccination strategy

Figure 3 represents the cost of the optimal use of u_1 and u_2. The curve initially increases to

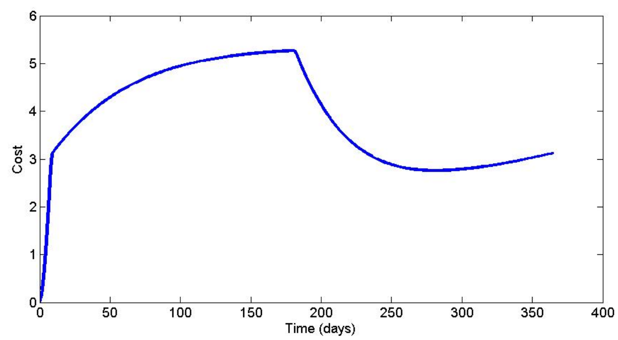
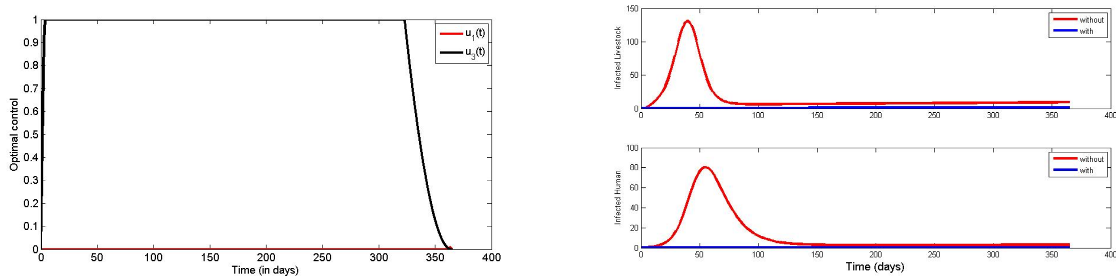


FIGURE 3. Optimal value for the cost of applying u_1 and u_2 strategy

near the maximal level because in the high infection level, more insecticides and vaccination doses will be needed which in turn represent the cost of intervention. The curve drops off to show that infection has decreased and hence the cost of intervention has also decreased.

3.2 Optimal insecticides and personal protection strategy

With this strategy, the control u_1 on mosquitoes and u_3 on humans are together used to optimise the objective function J, while u_2 is set to zero. Figure 4(a) shows that the control u_1 is nearly zero through out the time, while, the control u_3 is at its upper bound from $t = 4$ to $t = 322$ days before it falls up zero at the final time. The numerical results indicates that this strategy leaves more infecteds than it is in the first strategy, hence, suggesting that optimal use of insecticides and personal protection without vaccination is not effective. Figure 4(b) shows that there is a significant difference in the number of infecteds with and without control.



(a) optimal controls u_1 and u_3 (b) Infecteds with and without control

FIGURE 4. Optimal insecticides and personal protection strategy

Figure 5 shows the cost for the u_1 and u_3 strategy. The curve is an increasing function, which

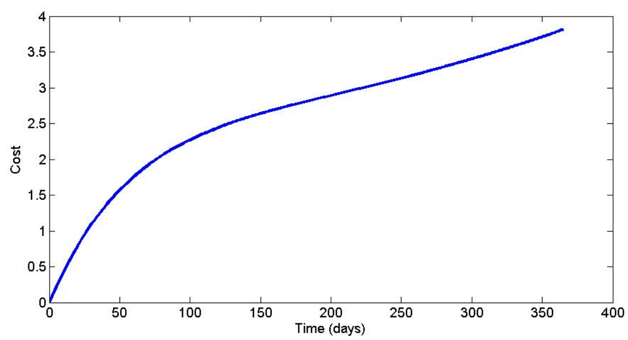
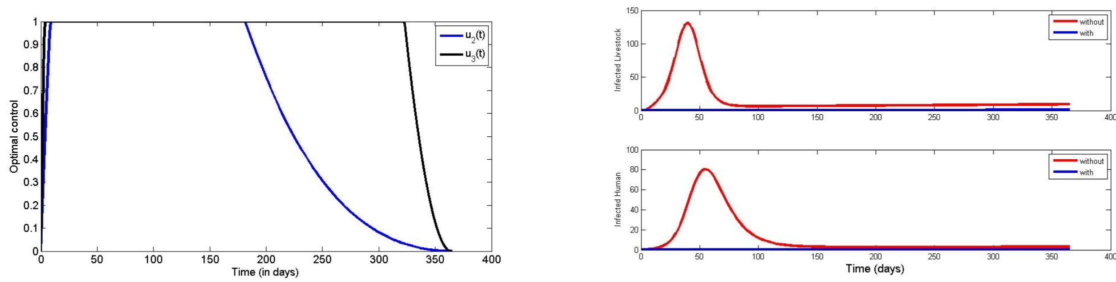


FIGURE 5. Optimal value for the cost of applying u_1 and u_3 strategy

indicates that the infection goes on increasing under this strategy.

3.3 Optimal livestock vaccination and personal protection strategy

With this strategy, the control u_2 on livestock and u_3 on humans are together used to optimise the objective function J, while u_1 is set to zero. Figure 6(a) shows that the control u_2 is at its upper bound from $t = 9$ to $t = 182$ days before it gradually fall down to zero, while, the control u_3 is at its upper bound from $t = 4$ to $t = 323$ days before it falls up zero at the final time. The numerical results indicates that this strategy gives similar results as the first strategy, suggesting that effective and optimal use of personal protection and vaccination without insecticides could also be beneficial. Figure 6(b) shows also that there is a significant difference in the number of infecteds with and without control.



(a) optimal controls u_2 and u_3

(b) Infecteds with and without control

FIGURE 6. Optimal livestock vaccination and personal protection strategy

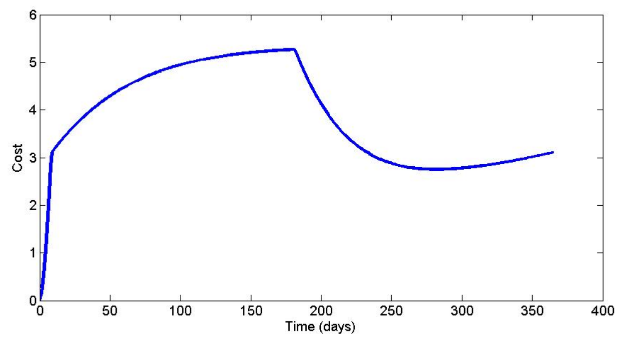
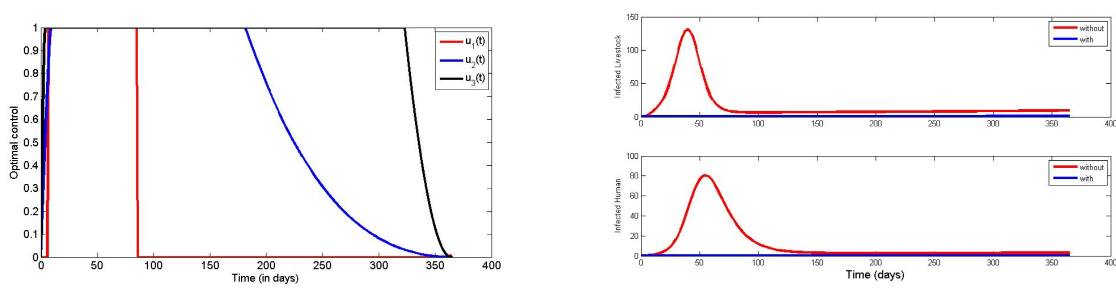


FIGURE 7. Optimal value for the cost of applying u_2 and u_3 strategy

Figure 7 shows the cost for administering u_2 and u_3 controls. The curve has similar characteristics as those in the first strategy.

3.4 Optimal insecticides, livestock vaccination, and personal protection strategy

Here all the controls are applied in order to optimise the objective function J. Figure 8(a)



(a) optimal controls u_1, u_2 and u_3

(b) Infecteds with and without control

FIGURE 8. Optimal strategy for applying both controls

shows the optimal control in applying all the strategies together. The graphs indicate that the control u_1 is at its upper bound from $t = 1$ to $t = 85$ days before rapidly fall to zero, the control u_2 is at its upper bound from $t = 9$ to $t = 182$ days before it gradually fall down to zero, while, the control u_3 is at its upper bound from $t = 4$ to $t = 323$ days before it falls to zero at the final time. The numerical results indicates that this strategy gives also similar results as the first strategy, suggesting that any combination strategy with vaccination is beneficial. Figure 8(b) shows the number of infecteds with and without control.

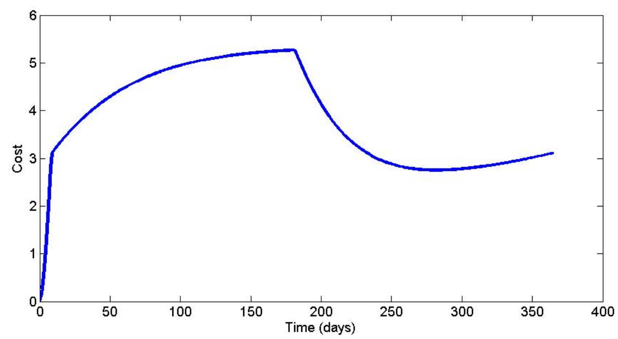


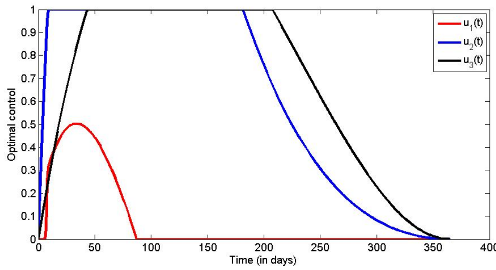
FIGURE 9. Optimal value for the cost of applying both controls

Figure 9 shows the cost for administering both controls in the same time horizon.

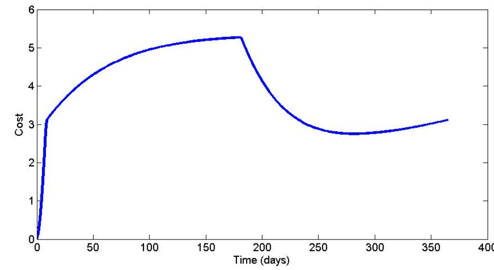
Further analysis shows that though the weights in the objective functional have influence on the shape of the control curves, they have little influence on the cost of the objective functional. Figure 10 shows the optimal trajectories for the controls and the cost of the objective functional for different set of weights in the objective functional.

4. Conclusion

In this article, we intended to assess the impact of some control measure in the dynamics of RVF. We derived and analysed the conditions for optimal control of RVF with insecticides, vaccination, and personal protection using optimal control theory. The results shows that the control measures has a very desirable effect for reducing the number of infected individuals and



(a) optimal controls u_1, u_2 and u_3



(b) Optimal value for the cost of applying both controls

FIGURE 10. Optimal strategy for both controls when $A_1 = A_2 = B_2 = 1000$, $B_1 = B_3 = 1$

that multiple controls are more effective than single control. Moreover, results shows that effective and optimal use of insecticides and personal protection without the use of vaccination is not beneficial if total elimination of the disease is desirable in the community. Any combination strategy which involve vaccination give better results and hence it may be beneficial.

Conflict of Interests

The authors declare that there is no conflict of interests.

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