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TWO-STEP ITERATIVE PROCEDURE FOR NON-EXPANSIVE MAPPINGS ON HADAMARD MANIFOLDS

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Abstract: In 2010, Chong Li et al. [14] studied the convergence of Mann and Halpern iterative procedures to a fixed point for non-expansive mappings on Hadamard manifolds. In this paper, inspired by Chong Li et al. [14], we study the convergence of Ishikawa iterative procedure (a two-step procedure) for non-expansive mappings on Hadamard manifolds. Moreover, the main result has been supported by a well-constructed example.

Keywords: Mann iterative procedure, Ishikawa iterative procedure, Halpern iterative procedure, non-expansive mappings, Hadamard manifolds.

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1. Introduction

Most of the results of non-expansive mappings have been obtained in normed linear spaces. The asymptotic behaviour of non-expansive mapping is, one of, the most active research areas in nonlinear functional analysis. But the most important analytical problem is the existence of fixed points for nonlinear mapping T, i.e., solutions of $x = T(x)$. Banach contraction principle states that the sequence of Picard iterates $\{T^n(x)\}$ converges strongly to a fixed point of T for any $x \in X$, if T is a contraction defined on a complete metric space X, one can find a huge literature about fixed points with different type of mappings [8, 13].

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In 1953, W. Robert Mann [15] introduced the most general iterative formula for approximation of fixed points of non-expansive mapping which is called Krasnoselskii–Mann iterative procedure. This procedure has been extensively studied by many authors [6, 10, 12, 18, 25]. Then, B. Halpern [9] gave an iterative procedure for a fixed point in 1967. Further in 1974, Ishikawa iteration procedure for approximating fixed points in Hilbert space has been introduced by S. Ishikawa [11]. In 1993, Tan and Xu [23] showed weak and strong convergence of Ishikawa iterative procedure for non-expansive mappings. H. Fukhar-ud-din [5] studied the strong convergence of Ishikawa iteration scheme for CAT(0) spaces. A CAT(0) space is simply a geodesic metric space whose each geodesic triangle is at least as thin as its comparison triangle in the Euclidean Plane. Further, Ishikawa iterative scheme has been studied extensively by many authors to solve the nonlinear equations in Hilbert space and Banach space [1, 3, 16, 17, 20, 21, 22, 26].

Recently, Chong Li et al. [14] studied the Mann and Halpern iterative algorithms for non-expansive mappings on Hadamard manifolds. Motivated by the results of Chong Li et al. [14], we study the Ishikawa iteration procedure for approximating a fixed point of non-expansive mappings in Hadamard manifolds (i.e. complete simply connected Riemannian manifolds of non-positive sectional curvature).

In this paper, our aim is to study the convergence of the Ishikawa iteration procedure (a two-step iterative procedure) to a fixed point for non-expansive mappings on Hadamard manifolds. The paper has been divided into three sections. Section 1 is of introductory nature. In Section 2, some basic concepts, results and notations on Riemannian manifolds have been given. In the last Section, we present our main result.

2. Preliminaries

First of all, we give some definitions and notations, which can be easily found in [2, 19].

Let $p \in M$, where M is a connected m-dimensional Riemannian manifold. A Riemannian manifold is a Riemannian metric $\langle ., . \rangle$, with the corresponding norm denoted by

$\| \cdot \|$. We denote the tangent space of M at p by T_pM. We define the length of a piecewise smooth curve, c: $[a, b] \to M$ joining p to q (i.e. $c(a) = p$ and $c(b) = q$), by using the metric as $L(c) = \int_a^b \|c'(t)\| \, dt$. Then the Riemannian distance $d(p, q)$ is defined to be the minimal length over the set of all such curves joining p to q, which induces the original topology on M. Let c be a smooth curve and ∇ be the Levi- Civita connection associated to $(M, \langle , \rangle)$. A smooth vector field X along c is said to be parallel if $\nabla_{c'} X = 0$. If c' is parallel, then c is a geodesic and here $\|c'\|$ is a constant. A geodesic joining p to q in M is said to be minimal geodesic if its length equals $d(p, q)$.A geodesic triangle $\Delta(p_1, p_2, p_3)$ of a Riemannian manifold is a set consisting of three points p_1, p_2 and p_3 and three minimal geodesic γ_i joining p_i to p_{i+1} , with $i = 1, 2, 3 (mod\ 3)$.

A Riemannian manifold is complete if for any $p \in M$, all geodesics emanating from p are defined for all $-\infty < t < \infty$. By the Hopf - Rinow theorem we know that if M is complete then any pair of points in M can be joined by a minimizing geodesic. Thus (M, d) is a complete metric space, and bounded closed subsets are compact.

Now, the exponential map $exp_p : T_pM \to M$ at $p \in M$ is such that $exp_p v = \gamma_v(1, p)$ for each $v \in T_pM$, where $\gamma(\cdot) = \gamma_v(\cdot, p)$ is the geodesic starting at p with velocity v. Then $exp_p\, tv = \gamma_v(t, p)$ for each real number t.

Definition 2.1 [19] A complete simply connected Riemannian manifold of non-positive sectional curvature is called a Hadamard Manifold.

Now, we present some basic results. We assume that M is a m-dimensional Hadamard manifold.

Proposition 2.1 [19] Let $p \in M$. Then $exp_p : T_pM \to M$ is a diffeomorphism, and for any two points $p, q \in M$ there exists a unique normalized geodesic joining p to q, which is in fact a minimal geodesic. This result shows that M has the topology and differential structure similar to $\mathbb{R}^m$. Thus Hadamard manifolds and Euclidean spaces have some similar geometrical properties.

Proposition 2.2 [19] (comparison theorem for triangles). Let Δ (p_1, p_2, p_3) be a geodesic triangle. For each i = 1, 2, 3(mod3), by $\gamma_i : [0, l_i] \to M$ the geodesic joining p_i to p_{i+1}, and set $l_i = L(\gamma_i)$, $\alpha_i = \angle(\gamma_i'(0) - \gamma_{i-1}'(l_{i-1}))$. Then

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \pi , \tag{2.1}$$

$$l_i^2 + l_{i+1}^2 - 2 l_i l_{i+1} \cos\alpha_{i+1} \leq l_{i-1}^2 . \tag{2.2}$$

In terms of the distance and the exponential map, the inequality (2.2) can be rewritten as

$$d^2(p_i, p_{i+1}) + d^2(p_{i+1}, p_{i+2}) - 2\left\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \right\rangle \leq d^2(p_{i-1}, p_i) , \tag{2.3}$$

since $\left\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \right\rangle = d(p_i, p_{i+1}) d(p_{i+1}, p_{i+2}) \cos\alpha_{i+1}$.

Proposition 2.3 [19] A subset $K \subseteq M$ is said to be convex if for any two points p and q in K, the geodesic joining p to q is contained in K, i.e., if $\gamma : [a, b] \to M$ is a geodesic such that $p = \gamma(a)$ and $q = \gamma(b)$, then $\gamma((1 - t)\, a + t\, b) \in K$ for all $t \in [0, 1]$. From now K will denote a nonempty, closed and convex set in M.

A real valued function f defined on M is said to be convex if for any geodesic γ of M, the composition function $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is convex, that is,

$$(f \circ \gamma)(t\, a + (1- t) b) \leq t\, (f \circ \gamma)(a) + (1-t)(f \circ \gamma)(b) \text{ for any } a, b \in \mathbb{R}, \text{ and } 0 \leq t \leq 1 .$$

Proposition 2.4 [19] Let d: $M \times M \to \mathbb{R}$ be a distance function. Then d is a convex function with respect to the product Riemannian metric, i. e., given any pair of geodesics γ_1: $[0,1] \to M$ and γ_2: $[0,1] \to M$ the following inequality holds for all $t \in [0,1]$:

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t) d(\gamma_1(0), \gamma_2(0)) + t d(\gamma_1(1), \gamma_2(1))$$

In particular, for each p $\in$ M, the function $d(\cdot, p)$: $M \to \mathbb{R}$ is a convex function.

Let P_K denote the projection onto K defined by

$$P_K(p) = \{ p_0 \in K : d(p, p_0) \leq d(p, q) \text{ for all } q \in K \}, \text{ for all } p \in M .$$

Proposition 2.5 [24] For any point $p \in M$, $P_K(p)$ is a singleton and the following inequality holds for all $q \in K$:

$$\left\langle \exp_{P_K(p)}^{-1} p, \exp_{P_K(p)}^{-1} q \right\rangle \leq 0 .$$

Definition 2.2 [4] Let X be a complete metric space and $F \subset X$ be a nonempty set. A sequence $\{x_n\} \subset X$ is called Fejer convergent to F if for all $y \in F$ and $n \geq 0$,

$$d(x_{n+1}, y) \le d(x_n, y).$$

Lemma 2.1 [4] Let X be a complete metric space. If $\{x_n\} \subset X$ is a Fejer convergent to a nonempty set $F \subset X$, then $\{x_n\}$ is bounded. Moreover, if a cluster point x of $\{x_n\}$ belongs to F, then $\{x_n\}$ converges to x.

Definition 2.3 [10, 11] Let $x_0 \in X$ be arbitrary. If the sequence $\{x_n\}_{n=0}^{\infty}$ satisfies the condition

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n$$

$$y_n = \beta_n x_n + (1-\beta_n)Tx_n \qquad (2.4)$$

for n = 0, 1, 2, 3,….., then this is called the Ishikawa iteration , where $\{\alpha_n\}$and $\{\beta_n\}$ are the sequences of positive numbers that satisfy the following condition:

(i) $0 \le \{\alpha_n\}, \{\beta_n\} \le 1$, for all positive integers n,

(ii) $\lim \alpha_n = 0$,

(iii) $\sum \alpha_n \beta_n = \infty$.

In 1993, K. K. Tan and H. K. Xu [23] established the result for non-expansive mapping $T : K \to K$, where K is a bounded closed subset of a uniformly convex Banach space X, that if

$$\sum_{n=0}^{\infty} \alpha_n (1-\alpha_n) = \infty \ , \ \sum_{n=0}^{\infty} \beta_n (1-\alpha_n) < \infty \ \text{ and } \ \lim_{n\to\infty} \beta_n < 1 \qquad (2.5)$$

then, $\|Tx_n - x_n\| \to 0$ as $n \to \infty$ which implies the convergence of $\{x_n\}$ to a fixed point of T if the range of T lies in a compact subset of X. In 1984, Goebel and Reich [7] studied the behaviour of the sequence of iterates $x_{n+1} = T(x_n)$ in Hyperbolic metric spaces.

In the next section, we study the convergence of Ishikawa iteration for non-expansive mappings in Hadamard manifold. Further, we consider the Ishikawa iteration in Hadamard manifolds M as follows:

$$x_{n+1} = \exp_{x_n}(1-\alpha_n)\exp_{x_n}^{-1} T(y_n), \quad y_n = \exp_{x_n}(1-\beta_n)\exp_{x_n}^{-1} T(x_n) \qquad (2.6)$$

for all $n \ge 0$, where $0 \le \{\alpha_n\}, \{\beta_n\} \le 1$.

3. Main results

In our main result, we will assume that M is a m-dimensional Hadamard Manifold.

Theorem 3.1 *Let K be a closed convex subset of M and $T : K \to K$ a non-expansive mapping with $F = Fix(T) \neq \emptyset$. Suppose that $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy the condition (2.5). Let $x_0 \in M$ and let $\{x_n\}$ be the sequence generated by the algorithm (2.6). Then $\{x_n\}$ converges to a fixed point of T.*

Proof: We know that K is a closed convex subset of M, thus K is a complete metric space. Using Lemma 2.1, it is sufficient to prove that $\{x_n\}$ is Fejer convergent to F and that all cluster points of $\{x_n\}$ belong to F. Now we suppose that $n \geq 0$ and $p \in F$ be fixed and γ_1 and γ_2 denote the geodesic joining x_n to $T(y_n)$ and y_n to x_n. Then $x_{n+1} = \gamma_1(1-\alpha_n)$ and $y_n = \gamma_2(1-\beta_n)$.

Now using the convexity of distance function and the non-expansivity of T, we have

$$d(x_{n+1}, p) = d(\gamma_1(1-\alpha_n), p) \leq \alpha_n d(x_n, p) + (1-\alpha_n) d(\mathrm{Ty}_n, p)$$

$$\leq \alpha_n d(x_n, p) + (1-\alpha_n) d(\mathrm{y}_n, p) \tag{3.1}$$

and $$d(\mathrm{y}_n, p) = d(\gamma_2(1-\beta_n), p) \leq \beta_n d(x_n, p) + (1-\beta_n) d(\mathrm{Tx}_n, p)$$

$$\leq \beta_n d(x_n, p) + (1-\beta_n) d(\mathrm{x}_n, p)$$

$$\Longrightarrow \quad d(y_n, p) \leq d(x_n, p) \tag{3.2}$$

By (3.1) and (3.2), we obtain

$$d(x_{n+1}, p) \leq \alpha_n d(x_n, p) + (1-\alpha_n) d(x_n, p) \leq d(x_n, p)$$

Hence $\{x_n\}$ is a Fejer convergent to F. Suppose x be a cluster point of $\{x_n\}$. Then there exists a subsequence $\{n_k\}$ of n such that $x_{n_k} \to x$. Now we prove that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.3}$$

For this, let $p \in F$ and $n \geq 0$. Let $\Delta(x_n, q, p)$ be the geodesic triangle with vertices x_n, $q = Ty_n$ and p. From Lemma 3.3 [14, p. 546] there exists a comparison triangle $\Delta(x_n', q', p')$ which conserves the length of edge. Also we have $x_{n+1} = \gamma_1(1-\alpha_n)$. Set

$x'_{n+1} = \alpha_n x'_n + (1-\alpha_n)Ty'_n = \alpha_n x'_n + (1-\alpha_n)q'$ as its comparison point. By Lemma 3.5(2) [14, p. 547].

$$d^2(x_{n+1}, p) \le \left\| x'_{n+1} - p' \right\|^2 = \left\| \alpha_n (x'_n - p') + (1-\alpha_n)(q' - p') \right\|^2$$

$$= \alpha_n \left\| x'_n - p' \right\|^2 + (1-\alpha_n) \left\| q' - p' \right\|^2 - \alpha_n (1-\alpha_n) \left\| x'_n - q' \right\|^2$$

$$= \alpha_n d^2(x_n, p) + (1-\alpha_n) d^2(Ty_n, p) - \alpha_n (1-\alpha_n) d^2(x_n, Ty_n)$$

$$\le \alpha_n d^2(x_n, p) + (1-\alpha_n) d^2(y_n, p) - \alpha_n (1-\alpha_n) d^2(x_n, Ty_n) \qquad (3.4)$$

Now, let $\Delta(x_n, \ell, p)$ be the geodesic triangle with vertices x_n, $\ell = Tx_n$ and p. From Lemma 3.3 [14, p. 546] there exists a comparison triangle $\Delta(x'_n, \ell', p')$ which conserves the length of edge. Also we have $y_n = \gamma_2(1\text{-}\beta_n)$ and set $y' = \beta_n x'_n + (1-\beta_n)Tx'_n == \beta_n x'_n + (1-\beta_n)\ell'$

Similarly, we can obtain

$$d^2(y_n, p) \le \| y'_n - p' \|^2 = \| \beta_n (x'_n - p') + (1-\beta_n)(\ell' - p') \|^2$$

$$\begin{aligned} &= \beta_n \| x'_n - p' \|^2 + (1-\beta_n) \| \ell' - p' \|^2 - \beta_n (1-\beta_n) \| x'_n - \ell' \|^2 \\ &= \beta_n d^2(x_n, p) + (1-\beta_n) d^2(Tx_n, p) - \beta_n (1-\beta_n) d^2(x_n, Tx_n) \\ &\le \beta_n d^2(x_n, p) + (1-\beta_n) d^2(x_n, p) - \beta_n (1-\beta_n) d^2(x_n, Tx_n) \\ &\le d^2(x_n, p) - \beta_n (1-\beta_n) d^2(x_n, Tx_n) \qquad (3.5) \end{aligned}$$

Combining (3.4) and (3.5), we obtain

$$d^2(x_{n+1}, p) \le \alpha_n d^2(x_n, p) + (1-\alpha_n)[d^2(x_n, p) - \beta_n (1-\beta_n) d^2(x_n, Tx_n)] - \alpha_n (1-\alpha_n) d^2(x_n, Ty_n)$$

$$\le \alpha_n d^2(x_n, p) + (1-\alpha_n) d^2(x_n, p) - \beta_n (1-\alpha_n)(1-\beta_n) d^2(x_n, Tx_n) - \alpha_n (1-\alpha_n) d^2(x_n, Ty_n)$$

$$\le d^2(x_n, p) - \beta_n (1-\alpha_n)(1-\beta_n) d^2(x_n, Tx_n) - \alpha_n (1-\alpha_n) d^2(x_n, Ty_n)$$

It follows that

$$\alpha_n (1-\alpha_n) d^2(x_n, Ty_n) + \beta_n (1-\alpha_n)(1-\beta_n) d^2(x_n, Tx_n) \le d^2(x_n, p) - d^2(x_{n+1}, p)$$

and

$$\sum_{n=1}^{\infty} \alpha_n (1-\alpha_n) d^2(x_n, Ty_n) < \infty \text{ and } \qquad \sum_{n=1}^{\infty} \beta_n (1-\alpha_n)(1-\beta_n) d^2(x_n, Tx_n) < \infty \qquad (3.6)$$

which implies that $\liminf_{n\to\infty} d(x_n, Ty_n) = 0$ and $\liminf_{n\to\infty} d(x_n, Tx_n) = 0$ (3.7)

because otherwise $d(x_n,T(x_n)) \geq a$ and $d(x_n,T(y_n)) \geq b$ for all $n \geq 0$ and for some a ,$b > 0$ and then

$$\sum_{n=1}^{\infty} \alpha_n (1-\alpha_n) d^2(x_n, Ty_n) \geq b \sum_{n=1}^{\infty} \alpha_n (1-\alpha_n) = \infty$$

and

$$\sum_{n=1}^{\infty} \beta_n (1-\alpha_n)(1-\beta_n) d^2(x_n, Tx_n) \geq a \sum_{n=1}^{\infty} \beta_n (1-\alpha_n) < \infty$$

which is a contradiction with(3.6).

On the other hand, the non-expansivity of T and convexity of the distance function, implies that

$$\begin{aligned} d(x_{n+1}, T(x_{n+1})) &\leq d(x_{n+1}, T(x_n)) + d(T(x_n), T(x_{n+1})) \\ &\leq d(x_{n+1}, T(x_n)) + d(x_n, x_{n+1}) \\ &\leq \alpha_n d(x_n, T(x_n)) + (1-\alpha_n) d(T(y_n), T(x_n)) \\ &\leq \alpha_n d(x_n, T(x_n)) + (1-\alpha_n) d(T(y_n), x_n) + (1-\alpha_n) d(y_n, T(x_n)) \end{aligned}$$

Now,

$$\begin{aligned} d\left(Ty_n, x_n\right) &\leq \beta_n d\left(x_n, x_n\right) + (1-\beta_n) d\left(Tx_n, x_n\right) \\ &\leq \beta_n d\left(x_n, x_n\right) + (1-\beta_n) d\left(x_n, x_n\right) \\ &\leq d\left(x_n, x_n\right) \end{aligned}$$

Therefore, $d(x_{n+1}, T(x_{n+1})) \leq \alpha_n d(x_n, T(x_n)) + (1-\alpha_n) d(x_n, T(x_n)) \leq d(x_n, T(x_n))$.

This means that $\{d(x_n, T(x_n)\}$is a monotone sequence. Combining this and (3.7) we get that (3.3) holds. Then, since

$$\begin{aligned} d(x, T(x)) &\leq d(x, x_{n_k}) + d(x_{n_k}, T(x_{n_k})) + d(T(x_{n_k}), T(x)) \\ &\leq 2d(x_{n_k}, x) + d(x_{n_k}, T(x_{n_k})) \end{aligned}$$

by taking limit, we deduce that $d(x, T(x)) = 0$, which means that $x \in \mathrm{Fix}(T)$.

Example: *Let* $X = H^3$ *and let* $N : X \to X$ *be a non-expansive mapping defined by* $N(x) = (-x_1, -x_2, -x_3, x_4)$ *for any* $x = (x_1, x_2, x_3, x_4) \in H^3$. *Also Fix (N) = (0, 0, 0, 1).*

To study the convergence to a fixed point for the non-expansive mapping $N : X \to X$ in the above example the following algorithms for Mann, Ishikawa and Halpern Iterative procedures have been taken:

Mann's algorithm [14]:

$$x_{n+1} = (\cosh(1-\alpha_n)r(x_n, x_n))\, x_n + (\sinh(1-\alpha_n)r(x_n,x_n))\, V(x_n, x_n), \text{ for all } n \geq 0,$$

Ishikawa's algorithm (2.6):

$$y_n = (\cosh(1-\beta_n)r(x_n, x_n))\, x_n + (\sinh(1-\beta_n)r(x_n,x_n))\, V(x_n, x_n),$$

$$x_{n+1} = (\cosh(1-\alpha_n)r(x_n, y_n))\, x_n + (\sinh(1-\alpha_n)\, r(x_n,y_n))\, V(x_n, y_n), \text{ for all } n \geq 0,$$

Halpern's algorithm [14]:

$$x_{n+1} = (\cosh(1-\alpha_n)r(u, x_n))\, u + (\sinh(1-\alpha_n)\, r(u,x_n))\, V(u, x_n) \text{for all } n \geq 0,$$

where $r(x, y) = arccosh(-\langle x, T(y)\rangle)$, $V(x, y) = \frac{T(y)+\langle x,T(y)\rangle x}{\sqrt{\langle x,T(y)\rangle^2-1}}$ for all $x, y \in H^m$

and $\alpha_n = \beta_n = \frac{1}{n+4}$ for each n = 0, 1, 2, 3, ……..

Following table denotes the error rates, where $e_n = d(x_n, z)$ is the error of nth step where $d(x, y) = \text{arccosh}(-\langle x, y\rangle)$ for all $x, y \in H^m$ and $z = (0, 0, 0, 1)$ is the unique fixed point and u = (0.60379247919382, 0.27218792496996, 0.19881426776106, 1.21580374135624) have been taken as fixed point for Halpern algorithm. Further, we take two random initial points x_0^1 and x_0^2 as follows:

x_0^1= (0.69445440978475, 1.01382609280137, 0.99360871330745, 1.87012527625153)

x_0^2 = (0.82054041398189, 1.78729906182707, 0.11578260956854, 2.20932797928782)

Table: 1

Errors for point x_0^1			
Steps	**Mann**	**Ishikawa**	**Halpern**
e_1	1.238499367948	1.238499367948	1.238499367948
e_2	0.619249683974	0.774062104968	0.791090879586
e_3	0.371549810384	0.526362231378	0.729379991797
e_4	0.247697396590	0.380148161064	0.516437319084
e_5	0.185842402714	0.297070596847	0.514416478068
e_6	0.139381802035	0.232086403786	0.382171112485
e_7	0.108408377987	0.186242577016	0.396372518479
e_8	0.086726702389	0.152718913153	0.302265746849
e_9	0.070958053361	0.127475890405	0.321961015335
e_{10}	0.059132184188	0.108001115351	0.249616001343
e_{11}	0.050035288972	0.092664241226	0.270836837639
e_{12}	0.042887247590	0.080367115303	0.212360403167
e_{13}	0.037168661996	0.070367350139	0.233579467968
e_{14}	0.032522579247	0.062122176982	0.184643326537
e_{15}	0.028696623024	0.055244476864	0.205239054320
e_{16}	0.025507854274	0.049446996693	0.163227538102
e_{17}	0.022822897533	0.044516200256	0.182965493183
e_{18}	0.020540607779	0.040287147532	0.146200420761
e_{19}	0.018584320295	0.036632860747	0.162810247457
e_{20}	0.016895005579	0.033454206192	0.132134829164
e_{21}	0.015425815895	0.030671440487	0.148058316136
Errors for point x_0^2			
e_1	1.430164437801	1.430164437801	1.430164437801
e_2	0.715082218901	0.893852773626	0.943247111267
e_3	0.429049331340	0.607819886065	0.836919431831
e_4	0.286030027231	0.438978327417	0.612255206541
e_5	0.214602608831	0.329356659494	0.590283208696
e_6	0.160951956623	0.257672415489	0.453503458308
e_7	0.125185212822	0.206926053024	0.453159103719
e_8	0.100148170258	0.169747941176	0.357434172184
e_9	0.081939229942	0.141723241962	0.367000303541
e_{10}	0.068283237879	0.120088612011	0.294325414299

e_{11}	0.057778544564	0.103044042605	0.307968375462
e_{12}	0.049524301688	0.089379450632	0.249929200461
e_{13}	0.042920731301	0.078258789021	0.265189980401
e_{14}	0.037555639888	0.069089303028	0.216850967037
e_{15}	0.033137594412	0.061440521948	0.232594780016
e_{16}	0.029455344921	0.054993019258	0.191345835389
e_{17}	0.026354875314	0.049509338352	0.207024100397
e_{18}	0.023719387783	0.044806060111	0.171102961798
e_{19}	0.021460353290	0.040741979059	0.186444990836
e_{20}	0.019509607176	0.037206852491	0.154658480978
e_{21}	0.017802516548	0.034112000043	0.169534343224

Fig. 1

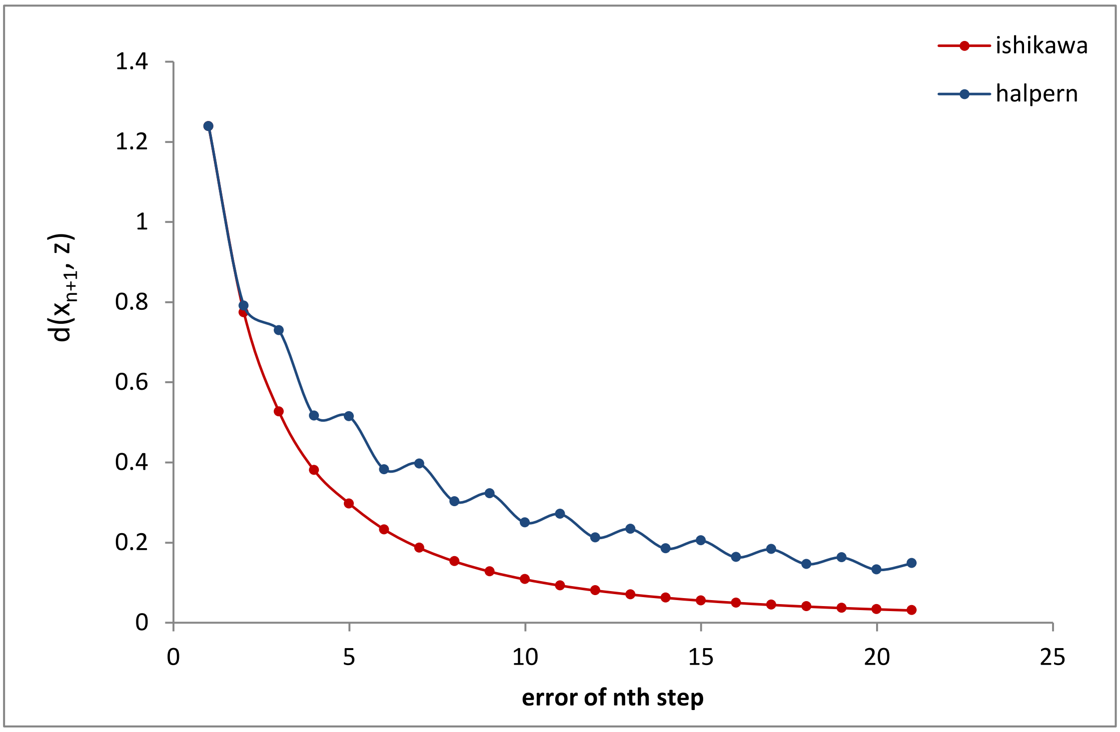
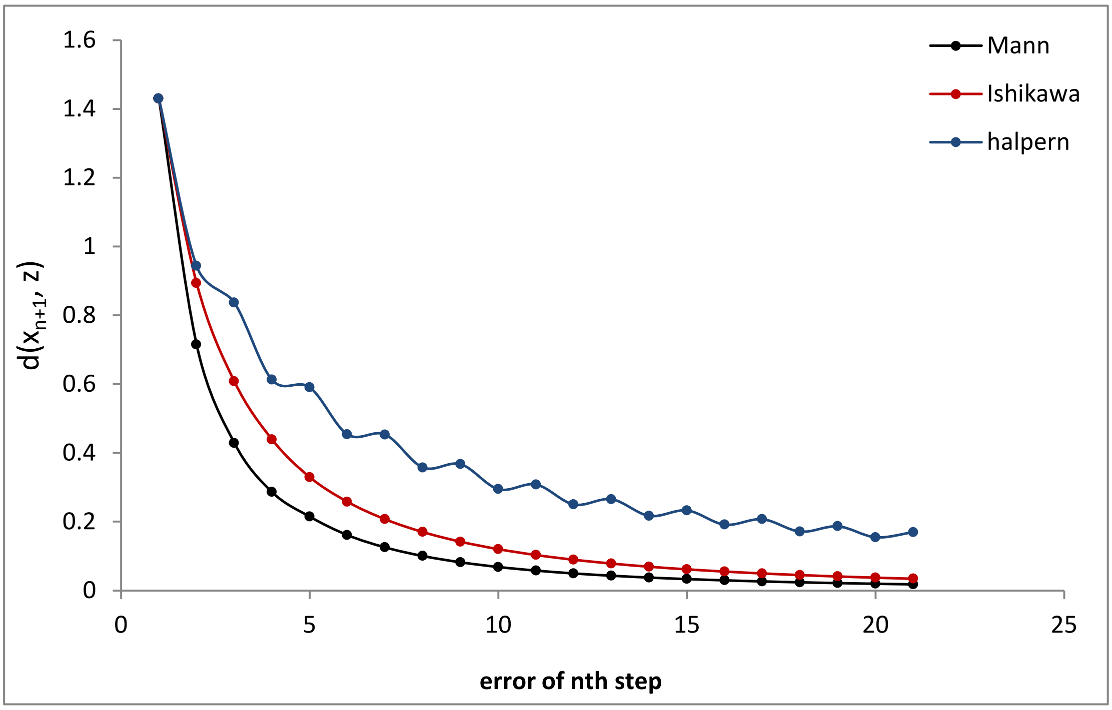


Fig. 2



4. Conclusion

In the computational study, a program has been generated in Mathematica to study the convergence to a fixed point for the non-expansive mappings using Mann, Ishikawa and Halpern iterative procedures and the following have been analyzed:

- Table 1, shows the outcomes of error rates produced by Mann, Ishikawa and Halpern iterative procedures for the random initial points x_0^1 and x_0^2.
- In Fig. 1,we analyzed that the Ishikawa iterative procedure gives lesser error rates than the Halpern iterative procedure, i.e., the convergence rate to a fixed point for non-expansive mapping is faster than the convergence rate of Halpern iterative procedure.(see also Fig. 2)

Conflict of Interests

The author declares that there is no conflict of interests.

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