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# OPTIMIZATION RELATED TO THE MARKOV MOMENT PROBLEM AND BOUNDS CONCERNING NEWTON'S METHOD FOR OPERATORS

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**Abstract:** The first aim of this work is to give necessary and sufficient conditions on the existence of a Markov moment problem in the first quadrant in terms of quadratic forms. This allows establishing the constraints for the related optimization problems. The second part of the paper contains evaluations related to the $p$ - root of a positive selfadjoint operator, $p \in R$, $p > 1$. The basic tools for this part are the contraction principle and a variant of Newton's method for convex operators. Some related bounds involving norms of operators are deduced.

**Keywords:** moment problem; quadratic forms; optimization; bounds

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## 1. Introduction

The first aim of this paper is to give a characterization for the existence of the solution of a classical multidimensional Markov moment problem in terms of (computable) quadratic forms. As it is well known, in several variables there are positive polynomials on the first quadrant that are not writable by means of sums of

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squares of some other polynomials. We show that $L^1-$ approximation results by sums of tensor products of positive polynomials in each separate variable on the positive semiaxes holds. For such polynomials, one knows the representation by means of sums of squares [1]. These results allow establishing the constraints in the associated optimization problem. A similar result on approximation applied to the complex moment problem appears in [18]. Application of extension results of linear operators and approximation to the moment problem appear in [1], [6], [9], [10], [16] – [19], [21], [23], [24], [28]. An introduction to optimization aspects of the moment problem is considered in [8] and [14]. For some other optimization problems see [12]. Other related interesting aspects on positive polynomials and their connection to the moment problem appear in [4] - [10], [18], [19], [21], [23] – [25], [28]. Uniqueness of the solution is considered in [1], [7], [13], [15], [29]. The background of this work is contained in [1], [8], [11], [14], [21], [26], [27].

The second part of this work contains approximation of $U^{1/p}, p>1, p\in R,$ where $U$ is a selfadjoint operator having the spectrum $\sigma(U)\subset[1,\infty).$

To this end, one uses the successive approximations method from the contraction principle and from Newton's method for convex operators [3], [22]. For local results concerning Newton's method for analytic operators, see [2]. The paper is organized as follows. Section 2 contains polynomial approximation results on unbounded subsets. The first part of Section 3 is devoted to establishing conditions for constrained optimization problem related to the moment problem. Upper and lower bounds are also considered. Secondly, one establishes bounds for $\left\|A-A^{1/p}\right\|, p>1, A$ being a positive selfadjoint operator.

## 2. Preliminaries

We recall the following approximation results on unbounded subsets of $R^n$.

**Lemma 2.1.** ([19], Lemma 1.3. (d)) *If* $x \in C_0([0,\infty))$ *is a nonnegative continuous function with compact support, then there exists a sequence* $(p_m)_m$ *of positive polynomials on* $[0,\infty)$, *such that*

$$p_m(t) > x(t)\ \forall t \geq 0,\ \forall m \in Z_+,\ p_m \to x$$

*uniformly on compact subsets of* $[0,\infty)$ *and in* $L^1_\nu([0,\infty))$, *for any M-determinate [1], [15] positive regular Borel measure [26]* $\nu$, *with finite moments of all natural orders.*

The next result is a generalization to several dimensions of approximation lemma 1.3 [19], point (a). Contrary to the preceding theorem, now the supports of the involved measures are not compact subsets.

**Lemma 2.2.** *The subset of sums tensor products* $p_1 \otimes p_2$ *of positive polynomials in separate variables* $t_1, t_2$ *is dense in* $\left(L^1_\nu\left([0,\infty)^2\right)\right)_+$, *for any measure* $\nu = \nu_1 \times \nu_2$, *where* $\nu_j,\ j = 1,2$ *are positive regular* M-*determinate Borel measures on* $R_+$, *with finite moments of all natural orders*.

**Proof.** Let $K_n = [0,n]\times[0,n],\ \phi \in \left(C\left([0,\infty)^2\right)\right)_+,\ \phi \equiv 0$ outside $K_n$. Due to Luzin and Weierstrass-Bernstein theorems, $\phi$ can be uniformly approximated on $K_n$ by a sums of tensor products $\psi_{1,n} \otimes \psi_{2,n}, \psi_{j,n} \in \left(C\left([0,n]\right)\right)_+,\ j = 1,2.$ Extend each function $\psi_{j,n}$ to $[0,\infty)$ such that the new functions vanish outside $[0,n]$. Approximate these new functions using Luzin's theorem, this time for functions of one variable. During this operation, the values of $\psi_{j,n}$ on $[0,n],\ j = 1,2,$ do not change. Thus, one obtains approximations with sums of tensor products of continuous nonnegative functions $\tilde{\psi}_{j,n},\ j = 1,2,\ n \in \mathrm{N},$ with compact support in separate variables. By Lemma 2.1 (see also lemmas 1.2, 1.3 [19]), there are positive polynomials

$$p_{j,m,n} = p_{j,m,n}\left(t_j\right),\ j=1,2,\ p_{j,m,n} \geq \tilde{\psi}_{j,n}\ \forall m,n \in \mathrm{N},$$
$$p_{j,m,n} \to \tilde{\psi}_{j,n}\ ,\ m \to \infty,\ j=1,2,\ n \in \mathrm{N}.$$

The convergence is uniform on compact subsets of $R_+$ and in $L^1_{\nu_j}\left(R_+\right)$. Then one can write:

$$\sum_{l=0}^{k(m,n)} p_{l,m,n} \otimes p_{l,m,n} \to \phi,\ m \to \infty,$$

in the norm of the space $L^1_\nu\left(R^2_+\right)$ and uniformly on $K_n$. Since this reasoning holds for any positive continuous function of compact support for a suitable $n \in \mathrm{N}$, and $\left(C_c\left(R^2_+\right)\right)_+$ is dense in $\left(L^1_\nu\left(R^2_+\right)\right)_+$, it follows that the cone generated by tensor products of positive polynomials in separate variables is dense in $\left(L^1_\nu\left(R^2_+\right)\right)_+$. The space generated by these tensor products is dense in $L^1_\nu\left(R^2_+\right)$. This concludes the proof. □

Positive Borel measures appear as representing positive linear functionals on the subspace of continuous functions of compact support, via Riesz representation theorem. For such measures, Luzin's theorem and approximation results by continuous functions with compact support work [26]. The assumption of being M – determinate means that a measure is uniquely determinate by its moments [1], [7], [15], [29]. For such measures, polynomial approximation from above holds for nonnegative continuous functions with compact support from $L^1_\nu\left(A\right), A \subset R^n$ being a closed and unbounded subset [19].

## 3. Main results

**Theorem 3.1.** (see [19], [23]) *Let* $\nu = \nu_1 \times \nu_2$ *be as in Lemma 2.2. Assume that* $\nu_1, \nu_2$ *are* $\sigma-$ *finite. Let* $\left(y_{(j,k)}\right)_{(j,k)\in \mathrm{N}^2}$ *be a sequence of real numbers. The following* *statements are equivalent:*

(a) *there is a unique function*

$$h \in L_\nu^\infty\left(R_+^2\right), 0 \le h\left(t_1, t_2\right) \le 1 \, a.e., \iint_{R_+^2} t_1^j t_2^k h\left(t_1, t_2\right) d\nu = y_{(j,k)}, \ \forall \left(j, k\right) \in \mathrm{N}^2;$$

(b) *for any finite subsets* $J_1, J_2 \subset \mathrm{N}$ *and all*

$$\left\{a_j\right\}_{j \in J_1} \subset R, \left\{b_m\right\}_{m \in J_2} \subset R,$$

*one has:*

$$0 \le \sum_{\substack{i,j \in J_1, \\ m,n \in J_2}} a_i a_j b_m b_n y_{(i+j, m+n)} \le \sum_{\substack{i,j \in J_1, \\ m,n \in J_2}} a_i a_j b_m b_n \iint_{R_+^2} t_1^{i+j} t_2^{m+n} d\nu$$

$$0 \le \sum_{\substack{i,j \in J_1, \\ m,n \in J_2}} a_i a_j b_m b_n y_{(i+j+1, m+n)} \le \sum_{\substack{i,j \in J_1, \\ m,n \in J_2}} a_i a_j b_m b_n \iint_{R_+^2} t_1^{i+j+1} t_2^{m+n} d\nu$$

$$0 \le \sum_{\substack{i,j \in J_1, \\ m,n \in J_2}} a_i a_j b_m b_n y_{(i+j, m+n+1)} \le \sum_{\substack{i,j \in J_1, \\ m,n \in J_2}} a_i a_j b_m b_n \iint_{R_+^2} t_1^{i+j} t_2^{m+n+1} d\nu$$

$$0 \le \sum_{\substack{i,j \in J_1, \\ m,n \in J_2}} a_i a_j b_m b_n y_{(i+j+1, m+n+1)} \le \sum_{\substack{i,j \in J_1, \\ m,n \in J_2}} a_i a_j b_m b_n \iint_{R_+^2} t_1^{i+j+1} t_2^{m+n+1} d\nu.$$

**Proof**. The implication $(a) \Rightarrow (b)$ is almost obvious. In order to prove the converse, one uses the density of sums of tensor products of positive polynomials in each one separate variable in the positive cone of $L_\nu^1\left(R_+^2\right)$ (Lemma 2.2). Using also the analytic expression of positive polynomials on the positive demiaxes [1], one observes that $(b)$ says that the linear form verifying the moment conditions is positive and dominated by $\nu$ on the cone generated by tensor products of positive polynomials in separate variables. In particular, this linear form has a positive extension $F,$ to the space of all integrable functions dominated in absolute value by a polynomial (see [11, p. 160]). The density proved in Lemma 2.1, leads to the continuity of this extension on the positive cone of the subspace of continuous functions of compact support. If $\phi$ is a nonegative continuous function with compact support, let

$$\sum_{l=0}^{k(m)} p_{l,1,m\otimes} p_{l,2,m} \to \phi,\ p_{k,m}(t_k) \geq 0,\ \forall t_k \geq 0,\ k = 1,2.$$

The convergence in the above formula is in the $L^1 -$ norm. The preceding relations and Fatou's lemma, as well as the hypothesis (b) yield:

$$F(\phi) \leq \liminf_m \sum_{l=0}^{k(m)} F\left(p_{l,m,1} \otimes p_{l,m,2}\right) \leq$$
$$\lim_m \sum_{l=0}^{k(m)} \iint_{R_+^2} \left(p_{l,m,1} \otimes p_{l,m,2}\right) d\nu = \iint_{R_+^2} \phi d\nu,\ \phi \geq 0.$$

For an arbitrary continuous function with compact support $\psi,$, one obtains:

$$\left|F(\psi)\right| \leq F\left(\psi^+\right) + F\left(\psi^-\right) \leq \iint_{R_+^2} |\psi| d\nu = \|\psi\|_1$$

Hence, $F$ is linear positive form, of norm at most one, on a dense subspace of $L_\nu^1\left(R_+^2\right)$. There is an extension preserving these qualities and a representing function $h \in L_\nu^\infty\left(R_+^2\right)$. It verifies the inequalities mentioned at point $(a)$ because of measure theory arguments. This concludes the proof. □

**Corollary 3.1.** *Under the hypothesis from above,* $\left(y_{j,k}\right)_{(j,k)\in N^2}$ *is a sequence of moments verifying (a) of Theorem 3.1 if and only if all the* $4-$ *forms*

$$\begin{aligned}
&\sum_{i,j=0}^{q} \left( \sum_{m,n=0}^{q} y_{i+j,m+n} \cdot a_i a_j b_m b_n \right) \geq 0, \\
&\sum_{i,j=0}^{q} \left( \sum_{m,n=0}^{q} \left(c_{i+j,m+n} - y_{i+j,m+n}\right) a_i a_j b_m b_n \right) \geq 0, \\
&\sum_{i,j=0}^{q} \left( \sum_{m,n=0}^{q} y_{i+j+1,m+n} a_i a_j b_m b_n \right) \geq 0 \\
&\sum_{i,j=0}^{q} \left( \sum_{m,n=0}^{q} \left(c_{i+j+1,m+n} - y_{i+j+1,m+n}\right) a_i a_j b_m b_n \right) \geq 0,
\end{aligned} \tag{3.1}$$

$$\sum_{i,j=0}^{q}\left(\sum_{m,n=0}^{q} y_{i+j,m+n+1}a_i a_j b_m b_n\right)\geq 0,$$
$$\sum_{i,j=0}^{q}\left(\sum_{m,n=0}^{q}\left(c_{i+j,m+n+1}-y_{i+j,m+n+1}\right)a_i a_j b_m b_n\right)\geq 0,$$
$$\sum_{i,j=0}^{q}\left(\sum_{m,n=0}^{q} y_{i+j+1,m+n+1}a_i a_j b_m b_n\right)\geq 0,$$
$$\sum_{i,j=0}^{q}\left(\sum_{m,n=0}^{q}\left(c_{i+j+1,m+n+1}-y_{i+j+1,m+n+1}\right)a_i a_j b_m b_n\right)\geq 0,\ q\in \mathrm{N},$$

*are semi positive definite for all* $\left(a=\left(a_j\right)_{j=0}^{q},\ b=\left(b_k\right)_{k=0}^{q}\right)$, *where*

$$c_{j,k}:=\iint_{R_+^2} t_1^j t_2^k d\nu,\ \left(j,k\right)\in \mathrm{N}^2.$$

The preceding result is important in itself and give the natural constraints for some optimization problems.

**Optimization Problem 1 (O. P. 1).** If

$$P\left(t_1,t_2\right)=\sum_{j,k=0}^{2q} c_{j,k}t_1^j t_2^k,\ c_{j,k}\in R,\left(j,k\right)\in N^2,\ q\in \mathrm{N}$$

is a given polynomial such that (a) of Theorem 3.1 is verified, then

$$\max(\min)\left(\iint_{R_+^2} P\left(t_1,t_2\right)h\left(t_1,t_2\right)d\nu;\ m_{j,k}\leq y_{j,k}\leq M_{j,k}\ \ \forall\left(j,k\right)\in \mathrm{N}^2\right)=$$
$$\max(\min)\left(\sum_{j,k=0}^{2q} c_{j,k}y_{j,k},\ m_{j,k}\leq y_{j,k}\leq M_{j,k}\ \ \forall\left(j,k\right)\in \mathrm{N}^2\right),\ j,k\leq q,\ s.t.\left(3.1\right).$$

Hence, we have a linear constrained programming problem. The maximum and minimum points are extreme points of the set defined by the restrictions (3.1) and by

$$m_{j,k}\leq y_{j,k}\leq M_{j,k},\quad \left(j,k\right)\in \mathrm{N}^2. \tag{3.2}$$

**Optimization problem 2 (O. P. 2):**

$$p_1(t_1)=\sum_{j=0}^{2q} a_j t_1^j,\ p_2(t_2)=\sum_{m=0}^{2q} b_m t_2^m \Rightarrow \max(\min)\left(\iint_{R_+^2}\left(p_1^2\otimes p_2^2\right)hd\nu\right)=$$
$$\max(\min)\left(\sum_{i,j=0}^{2q}\left(\sum_{m,n=0}^{2q} a_i a_j b_m b_n y_{i+j,m+n}\right)\right) s.t.\ (3.1), (3.2)\ j,k\le q.$$

This is also a constrained optimization problem with linear objective function.

**Lower and upper bounds (L - U. B.).** Assume that we have solved (O. P. 1). Let $\alpha = k/l,\ k,l\in \mathrm{N}\setminus\{0\}$ be a given positive not integer rational number, with even numerator. An application of Jensen's inequality leads to:

$$\frac{\iint_{R_+^2} P^{k/l}(t_1,t_2)h(t_1,t_2)d\nu}{\iint_{R_+^2} h(t_1,t_2)d\nu} \le \left(\frac{\iint_{R_+^2} P^{k}(t_1,t_2)h(t_1,t_2)d\nu}{\iint_{R_+^2} h(t_1,t_2)d\nu}\right)^{1/l}$$

Under the constraints (3.1), (3.2), the right hand size member can be majorized applying (O. P. 1). Thus, one obtains an upper bound for a nonlinear objective function, with a domain defined by suitable semi positive matrix. If $\alpha>1$ and $P$ is a convex positive polynomial on the convex hull of the feasible set corresponding to $P,$ one can find an upper bound for

$$\max\iint_{R_+^2} P^{\alpha}(t_1,t_2)d\nu\ s.t.\ (3.1), (3.2), \alpha>1,$$

To this end, one applies the maximum principle for the composed convex functional on this convex compact subset. If $\alpha$ is as above, but $\alpha\in(0,1),$ for an upper bound one applies either Jensen's inequality, or Hölder's inequality. For a lower bound, if the polynomial is concave and positive, one uses the minimum principle for the concave continuous function on the convex hull of the set defined by means of (3.1), (3.2).

The next results concern approximations and bounds related to $A^{1/p},\ p\in R, p>1,$

where $A$ is a selfadjoint operator on an arbitrary Hilbert space $H$, and the spectrum $\sigma(A)$ is contained in $(0,\infty)$. Firstly, we consider the $p-$ root of $A$ as the solution of the equation

$$P(U) = U^p - A = 0.$$

One can prove that $P$ is convex and increasing on the convex cone of selfadjoint positive operators. Hence the associated variant of Newton's method [3], [22] works. Namely, the following statements hold true, under some hypothesis.

Let X be a $\sigma-$ complete vector lattice, endowed with a solid $(|x| \le |y| \Rightarrow \|x\| \le \|y\|)$ and $o-$ continuous norm $(x_n \to_{in\ order} x \Rightarrow \|x_n - x\| \to 0)$. Let $Y$ be a normed vector space, endowed with an order relation defined by a closed convex cone. For $a,b \in X, a < b,$ we denote $[a,b] = \{x \in X, a \le x \le b\}$. Let $P \in C^1([a,b], Y)$. In most of our applications, we have $X = Y$.

**Theorem 3.2.** (see [3], [22]). *Assume additionally that for each*

$$x \in [a,b], \exists [P'(x)]^{-1} \in L_+(Y, X)$$

*and that*

$$a \le x \le b \Rightarrow P'(a) \le P'(x) \le P'(b).$$

*If* $P(a) < 0, P(b) > 0,$ *then there exists a unique solution* $x^*$ *of the equation* $P(x) = 0,$ *where*

$$x^* := \inf x_k = \lim x_k,\ x_0 := b,\ x_{k+1} = x_k - [P'(x_k)]^{-1}[P(x_k)],\ k \in N. \qquad (3.3)$$

*Moreover, we have*

$$a < x^* < b, \quad \|x_k - x^*\| \le \left\| \left[ P'(a) \right]^{-1} \right\| \cdot \|P(x_k)\| \to 0. \tag{3.4}$$

Let $H$ be a Hilbert space, $A$ a selfadjoint operator acting on $H$, with the spectrum

$$S(A) \subset ]0, \infty[, \; B \in \mathrm{A}(H), Y = Y(B)$$

the associated commutative algebra according to (3.5) from below:

$$Y_1 = Y_1(B) = \{T \in \mathrm{A}(H);\, BT = TB\}, Y = Y(B) = \{T \in Y_1;\, TU = UT, \;\; \forall U \in Y_1\}. \tag{3.5}$$

The space $Y$ is a commutative algebra, which is also an order complete Banach lattice, with solid norm (see [11], [17]). We denote

$$\omega_A = \inf_{\|h\|=1} \langle Ah, h \rangle, \quad \Omega_A = \sup_{\|h\|=1} \langle Ah, h \rangle$$

**Theorem 3.3.** (see Theorem 2.1 [3]). *Let* $A$ *be as above,* $A \notin Sp\{I\}$, $p > 1, p \in R.$ *There exists a unique operator* $U_p \in ]\omega_A^{1/p} I, \Omega_A^{1/p} I[$ *such that*

$$U_p^p - A = 0,$$

*and this solution verifies the relations*

$$\left\| \Omega_A^{1/p} I - U_p \right\| \le \frac{1}{p \omega_A^{(p-1)/p}} \|\Omega_A I - A\|.$$

**Remark 3.1.** If in the recurrence relation of Newton's method:

$$x_{k+1} = \phi(x_k), \phi(x) := x - \left[ P'(x) \right]^{-1} \left( P(x) \right)$$

the mapping $\phi$ is a contraction, the rate of convergence of the sequence $(x_k)_k$ is given by contraction principle. Next, we recall that this is the case of the operator $P(U) = U^p - A,$ which leads to the positive solution $U_p = A^{1/p}.$ The next result establish the connection between Newton's method and the successive approximation method of the contraction principle. Iterations given by (3.3) become (3.6) in this particular case. Relation (3.7) is a consequence of the corresponding evaluation from the contraction principle. It is not equivalent to an application of (3.4).

**Theorem 3.4.** (see Theorem 3.1 [3]). *Let* $p, A, X$ *be as above. Then the Newton recurrence for the equation*

$$P(U) = U^p - A = 0$$

*is*

$$U_0 = \Omega_A^{1/p} I, \quad U_{k+1} = \phi(U_k) = \frac{p-1}{p} U_k + \frac{1}{p} U_k^{-p+1} A, \quad k \in N. \tag{3.6}$$

*The convergence rate for* $U_k \to A^{1/p}$ *is given by*

$$\left\| U_k - A^{1/p} \right\| \le \left( \frac{p-1}{p} \right)^k \Omega_A^{1/p} \left\| I - \Omega_A^{-1} A \right\|, \quad k \in N. \tag{3.7}$$

Relation (3.7) gives the convergence rate of $\left\| U_k - A^{1/p} \right\| \to 0, k \to \infty$ as being smaller or equal to $O\left( \left( \frac{p-1}{p} \right)^k \right)$. The equalities (3.6) are exactly (3.3), written for our operator $P$ and initial value $U_0$ from the statement of Theorem 3.4.

**Corollary 3.2.** *If* $\sigma(A)\subset[1,\infty), A\notin Sp\{I\}$, *then we have*:

$$\left\|A-A^{1/p}\right\|\le\frac{1}{p\omega_A^{(p-1)/p}}\left\|A^p-A\right\|,\ \forall p>1.$$

**Proof.** The hypothesis on the spectrum of $A$ leads to $A\ge A^{1/p}, p>1.$ Thus, $A$ stands for the initial approximation $x_0=b$ from (3.3). Choosing

$$a=\omega_A^{1/p}\cdot I<A^{1/p},\ A\ne I,$$

the statement follows by an application of (3.4) to $P(A)=A^p-A.$ This concludes the proof. □

Secondly, in the next results we obtain upper bounds for some norms of operators involving $A^{1/p}$, directly from the contraction principle. Namely, we prove the following theorem, stated for operators in the commutative Banach algebra $Y(B)$ defined by (3.5).

**Theorem 3.5.** *Let* $C=\{U\in Y(B);\sigma(U)\subset[1,\infty)\}$ *and* $A\in C.$ *Then for all* $k\in\mathrm{N}, k\ge 1,$ *we have:*

$$(p-1)p^{k-1}\left\|A^{1/p^k}-I\right\|\le\left\|A-A^{1/p}\right\|,\ p>1. \tag{3.8}$$

*Consequently, the following relations hold*

$$(p-1)\limsup_k\left(p^{k-1}\left\|A^{1/p^k}-I\right\|\right)\le\left\|A-A^{1/p}\right\|,$$
$$\limsup_p\left((p-1)\left\|A^{1/p}-I\right\|\right)\le\left\|A-I\right\|,\ \forall A\in C.$$

**Proof.** For $U \in C$ and $p > 1, U^{1/p}$ have sense given by the analytic functional calculus. We define $Q : C \to C,$

$$Q(U) = U^{1/p}, \quad U \in C.$$

The operator $Q$ is a concave operator, applying $C$ into itself. Namely, if $\sigma(U) \subset [1,\infty),$ we have

$$\sigma\left(U^{1/p}\right) = \left(\sigma(U)\right)^{1/p} \subset [1,\infty).$$

The operator $Q$ is from $C$ onto $C.$ Next we show that $Q$ is a contraction from $C$ onto $C,$ the contraction constant being $(1/p) < 1.$ For $U, V \in C,$ we have:

$$Q'(U)(V) = \frac{1}{p} U^{\frac{1}{p}-1} V; \frac{1}{p} - 1 < 0 \Rightarrow$$
$$\sigma\left(U^{\frac{1}{p}-1}\right) = \left(\sigma(U)\right)^{1/p-1} \subset (0,1].$$

From the last relation we infer that

$$\langle U^{\frac{1}{p}-1} h, h\rangle \le \Omega_{U^{1/p-1}} \le 1 = \langle Ih, h\rangle,\ h \in H,\ \|h\| = 1 \Rightarrow U^{\frac{1}{p}-1} \le I,\ \forall U \in C.$$

Using the fact that the product of two positive commuting operators is positive, the preceding relations lead to

$$\left(I - U^{1/p-1}\right) \cdot V \ge 0 \Rightarrow Q'(U) \cdot V = \frac{1}{p} U^{\frac{1}{p}-1} V \le \frac{1}{p} V,\ \ \forall U, V \in C.$$

On the other hand, the following evaluations hold:

$$\|Q(U_1)-Q(U_2)\| \le \sup_{U\in C}\left\|\frac{1}{p}U^{\frac{1}{p}-1}\right\|\cdot\|U_1-U_2\| \le \frac{1}{p}\|U_1-U_2\|, U_1,U_2\in C.$$

Hence $Q$ is a contraction from the complete metric space $C$ onto $C,$ which has a unique fixed "point". This fixed point is the identity operator $I.$ From the well known evaluation given by contraction theorem, we have:

$$\|(Q\circ\cdots\circ Q)(U)-I\| \le \frac{(1/p^k)}{1-1/p}\|U-Q(U)\|, k\in \mathrm{N}.$$

for any initial approximation $U\in C$ of $I.$ The composition of $Q$ with itself occurs $k$ times in the above relation. We rewrite the above relation as

$$\left\|U^{1/p^k}-I\right\| \le \frac{1}{p^{k-1}(p-1)}\left\|U-U^{1/p}\right\|, \quad U\in C, k\in \mathrm{N}. \tag{3.9}$$

This proves (3.8). The other assertions of the statement follow. This concludes the proof. □

**Corollary 3.3.** *For all* $U\in C$ *and all* $p>1,$ *we denote:*

$$E(p,U)=\left[\left(\frac{\left\|U^{1/p^2}-I\right\|^2}{\left\|U-U^{1/p}\right\|^2}+4\frac{\left\|U^{1/p^2}-I\right\|}{\left\|U-U^{1/p}\right\|}\right)^{1/2} - \frac{\left\|U^{1/p^2}-I\right\|}{\left\|U-U^{1/p}\right\|}\right].$$

*Then the following relations hold:*

$$p\cdot E(p,U)\le 2, \limsup_p\left(p\cdot E(p,U)\right)\le 2, \quad \forall U\in C, U\ne I.$$

**Proof.** We write (3.9) for the case $k=2.$ One obtains:

$$(p-1)\cdot p\cdot\left\|U^{1/p^2}-I\right\|\le\left\|U-U^{1/p}\right\|,\ p>1, U\in C\setminus\{I\}\Leftrightarrow$$
$$\left\|U^{1/p^2}-I\right\|\cdot p^2-\left\|U^{1/p^2}-I\right\|\cdot p-\left\|U-U^{1/p}\right\|\le 0.$$

The signature of the two-degree polynomial with two real roots of opposite signatures in the variable $p$ and the fact that $p>1$ lead to the following relations

$$p\le\frac{\left\|U^{1/p^2}-I\right\|+\left(\left\|U^{1/p^2}-I\right\|^2+4\cdot\left\|U^{1/p^2}-I\right\|\cdot\left\|U-U^{1/p}\right\|\right)^{1/2}}{2\cdot\left\|U^{1/p^2}-I\right\|}=$$
$$2\frac{\left\|U-U^{1/p}\right\|}{\left[\left(\left\|U^{1/p^2}-I\right\|^2+4\cdot\left\|U^{1/p^2}-I\right\|\cdot\left\|U-U^{1/p}\right\|\right)^{1/2}-\left\|U^{1/p^2}-I\right\|\right]}.$$

Now the conclusion follows by dividing the numerator and the denominator of the last fraction by the same factor $\left\|U-U^{1/p}\right\|.$ This concludes the proof. □

**Corollary 3.4.** *If* $\sigma(A)\subset[1,\infty),\ A\notin Sp\{I\},\ p>1, k\in\mathrm{N}, k\ge 1,$ *then we have:*

$$(p-1)\,p^{k-1}\left\|A^{1/p^k}-I\right\|\le\left\|A-A^{1/p}\right\|\le\frac{1}{p\omega_A^{(p-1)/p}}\left\|A^p-A\right\|.$$

**Proof.** The relation (3.8) gives the first inequality, while the second one follows from the Corollary 3.2. This concludes the proof. □

### Conflict of Interests

The author declares that there is no conflict of interests.

REFERENCES

[1] N. I. Akhiezer, The Classical Moment Problem and some related Questions in Analysis, Oliver and Boyd, Edinburgh and London, 1965.

[2] I. K. Argyros, On the convergence and applications of Newton-like methods for analytic ope*rators*, J. Appl. Math. & Computing, 10**,** 1-2 (2002), 41-50.

[3] V. Balan, A. Olteanu and O. Olteanu, On Newton's method for convex operators, and some applications, Romanian Journal of Pures and Applied Mathematics, 51, 3 (2006), 277-290

[4] C. Berg, J. P. R. Christensen, C. U. Jensen, A remark on the multidimensional moment problem, Mathematische Annalen, 243 (1979), 163-169.

[5] C. Berg, P. H. Maserick, Polynomially positive definite sequences, Mathematische Annalen, 259 (1982), 487-495.

[6] C. Berg, J. P. R. Christensen, P. Ressel, Harmonic Analysis on Semigroups. Theory of Positive Definite and Related Functions, Springer, 1984.

[7] C. Berg, M. Thill, Rotation invariant moment problems, Acta Mathematica, 11 (1991), 207-227.

[8] S. Boyd and L. Vandenberghe, Convex optimization. Cambridge University Press, 2004, 2009.

[9] G. Cassier, Problèmes des moments sur un compact de $R^n$ et décomposition des polynômes à plusieurs variables, Journal of Functional Analysis, 58 (1984), 254-266.

[10] G. Choquet, Le problème des moments, Séminaire d'Initiation à l'Analise, Institut H. Poincaré, Paris, 1962.

[11] R. Cristescu, Ordered Vector Spaces and Linear Operators, Ed. Academiei, Bucharest, and Abacus Press, Tunbridge Wells, Kent, 1976.

[12] C. Drăguşin, Min-max pour des critères multiple, R. A. I. R. O., Operations Research 12**,** 2 (1978), 169-180.

[13] B. Fuglede, The multidimensional moment problem, Expo. Math. I (1983), 47-65.

[14] R. B. Holmes, Geometric Functional Analysis and its Applications. Springer-Verlag New York Inc., 1975.

[15] C. Kleiber and J. Stoyanov, Multivariate distributions and the moment problem, Journal of Multivariate Analysis, 113 (2013), 7-18.

[16] M. G. Krein and A. A. Nudelman, Markov Moment Problem and Extremal Problems, American Mathematical Society, Providence, RI, 1977.

[17] L. Lemnete-Ninulescu, Using the solution of an abstract moment problem to solve some classical complex moment problems, Rev. Roumaine Math. Pures Appl., 51 (2006), 703-711.

[18] L. Lemnete-Ninulescu and A. Zlătescu, Some new aspects of the L moment problem, Rev. Roumaine Math. Pures Appl., 55**,** 3 (2010), 197-204.

[19] A. Olteanu and O. Olteanu, Some unexpected problems of the Moment Problem, Proc. of the Sixth Congress of Romanian Mathematicians, Ed. Academiei, Vol I, Bucharest, (2009), 347-355.

[20] O. Olteanu, Convexité et prolongement d'opérateurs linéaires, C. R. Acad. Sci. Paris, 286, Série A (1978), 511-514.

[21] O. Olteanu, Application de théorèmes de prolongement d'opérateurs linéaires au problème des moments et à une généralisation d'un théorème de Mazur-Orlicz, C. R. Acad. Sci Paris 313, Série I (1991), 739-742.

[22] O. Olteanu and Gh. Simion, A new geometric aspect of the implicit function principle and Newton's method for operators. Math. Reports, 1 (2003), 61-84.

[23] O. Olteanu, New results on Markov moment problem, International Journal of Analysis, Vol. 2013, Article ID 901318, 17 pages, http://dx.doi.org/10.1155/2013/901318

[24] M. Putinar, Positive polynomials on compact semi-algebraic sets, Indiana University Mathematical. Journal, **42**, 3 (1993), 969-984.

[25] M. Putinar and F. H. Vasilescu, Problème des moments sur les compacts semi-algébriques, C. R. Acad Sci. Paris, **323**, Série I (1996), 787-791.

[26] W. Rudin, Real and complex analysis. Third edition, McGraw-Hill, Inc., 1987.

[27] H. H. Schaefer, Topological Vector Spaces, Springer, 1971.

[28] K. Schmüdgen, The $K-$ moment problem for compact semi-algebraic sets, Mathematische Annalen, 289 (1991), 203-206.

[29] J. Stoyanov, Stieltjes classes for moment indeterminate probability distributions, J. Applied Probab. 41 A (2004), 281-294.