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MARKOV MOMENT PROBLEM AND APPROXIMATION

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Abstract: In this paper, we start by recalling some polynomial approximation results on unbounded subsets of R^n. Namely one approximates nonnegative continuous functions with compact support by sums of tensor products of positive polynomials on the positive semiaxes, in each separate variable. For such polynomials, expression in terms of sums of squares is well known. This method leads to characterization of the existence of the solution of the multidimensional Markov moment problem in terms of quadratic mappings. Two other applications related to the Markov moment problem are considered. Here the main ingredients of the proofs are extension of linear operator's results, involving two constraints.

Keywords: convexity; moment problem; approximation

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1. Introduction

Polynomial approximation in studying existence and uniqueness of the solutions of some moment problems have been discussed in [1] - [18] and in many other works.

Using extension of linear operators in proving the existence of a solution for a moment problem is a well-known method. In the first part of this paper, we recall

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some results from [11], [12] concerning polynomial uniform approximation on compact subsets of unbounded closed subsets of R^n, and in L^1 norm too. This is not the case of L^2 norm, following [2]. An application to the multidimensional moment problem is considered. The general idea is to approximate some nonnegative continuous functions in two variables, on the first quadrant, by sums of tensor products of positive polynomials in each separate variable. The analytic expression for such polynomials by means of sums of squares is discussed in [1]. Thus, the difficulty of the existence of positive polynomials in several variables, which are not writable in terms of sums of squares seems to be solved (Section 3). Note that this method works for L^1 spaces too, for measures with unbounded support.

A similar approximation result used in solving the complex multidimensional moment problem appears in [9]. The second theme of the present paper contains applications of some extension results for linear operators, with two constraints, to the Markov moment problem (Section 3).

For the uniqueness of the solution see [2], [4], [5]. The background of this work is contained in [1], [19], [20]. One of our results on the abstract moment problem [13] generalizes a key theorem from [18] (see theorem 3.4 from below). The works [21], [22], deal with classes of special operators and approximation results.

The paper is organized as follows. Section 2 contains preliminaries on polynomial approximation in unbounded subsets. Section 3 (the main results), is devoted to an application of the approximation results to a two dimensional moment in the first quadrant. Two other Markov moment problem are solved, by using constrained extension theorems for linear operators. Using Hahn – Banach type results and their generalizations represents a common characteristic of this work.

2. Preliminaries

The following results were published in [11], [12], being applied in solving moment problems from [12], recently recalled in [15], [16]. For the multidimensional moment problem on unbounded subsets, Stone-Weierstrass and Luzin's theorems are used too. Thus, one characterizes the existence of solutions of multidimensional Markov moment problems in terms of quadratic mappings, similarly to the one-dimensional case (Section 3).

Lemma 2.1. *For any* $x \in R$, *we have*

$$\exp(x) - \left(1 + \frac{x}{1!} + \ldots + \frac{x^m}{m!}\right) = \frac{\exp(x)}{m!} \cdot \int_0^x \exp(-t) \cdot t^m dt, \; m \in \mathrm{N}.$$

The proof is quite standard. Multiplication by $\exp(-x)$ followed by derivation-operation leads to the equality of the derivatives. Then the conclusion follows easily.

Remark 2.1. The statement remains true when we replace $x \in R$ with $z \in C$, by analytic continuation.

Corollary 2.1. *For all* $\alpha > 0, k \in \mathrm{N}$ *we have*

$$\exp(-\alpha t) \leq 1 - \frac{\alpha \cdot t}{1!} + \frac{\alpha^2 \cdot t^2}{2!} - \cdots + \frac{\alpha^{2k} t^{2k}}{(2k)!}, \; t \geq 0,$$
$$\exp(-\alpha t) \geq 1 - \frac{\alpha \cdot t}{1!} + \frac{\alpha^2 \cdot t^2}{2!} - \cdots + \frac{\alpha^{2k} t^{2k}}{(2k)!} - \frac{\alpha^{2k+1} t^{2k+1}}{(2k+1)!}, \; t \geq 0.$$

Corollary 2.2. *Let* $\phi_k(t) = \exp(-kt), t \geq 0, k \in \mathrm{N},$ *and* ψ *an element of the linear subspace generated by* $\{\phi_k; k \in \mathrm{N}\}$. *Then there exists a sequence of polynomials*

$$(p_l)_{l \in \mathrm{N}}, \; p_l(t) > \psi(t) \; \forall t \geq 0,$$

and $\lim p_l = \psi$ *uniformly on the compact subsets* $K \subset [0, \infty)$ *and in* $L^1_\nu([0, \infty))$, *for any positive regular M – determinate [1], [5] Borel measure* ν *on* $[0, \infty)$, *with finite* *moments of all natural orders.*

Lemma 2.2. *Let* $\psi:[0,\infty)\to R_+$ *be a continuous function, such that* $\lim_{t\to\infty}\psi(t)\in R_+$ *exists. Then there is a decreasing sequence* $(h_l)_l$ *in the linear hull of the functions* $\phi_k, k\in \mathrm{N}$ *defined above, such that* $h_l(t)>\psi(t), t\ge 0, l\in \mathrm{N}, \lim h_l=\psi$ *uniformly on* $[0,\infty)$. *There exists a sequence* $(\tilde{p}_l)_l, \tilde{p}_l\ge h_l>\psi, \forall l\in \mathrm{N}, \lim \tilde{p}_l=\psi$ *uniformly on compact subsets of* $[0,\infty)$ *and in* $L^1_\nu([0,\infty))$, *where* ν *is as above.*

Positive Borel measures appear as representing positive linear functionals on the subspace of continuous functions of compact support, via Riesz representation theorem. For such measures, Luzin's theorem and approximation results by continuous functions with compact support work. The assumption of being M – determinate means that a measure is uniquely determinate by its moments [1], [2], [5]. For such measures, polynomial approximation from above holds for nonnegative continuous functions with compact support from $L^1_\nu(A), A\subset R^n$ being a closed and unbounded subset [11], [12].

Lemma 2.3. *Let* $\nu=\nu_1\times\nu_2$ *be a product of two determinate positive regular Borel measures on* R_+, *with finite moments of all natural orders. Then any nonnegative continuous function with compact support is approximated in* $L^1_\nu(R^2_+)$ *and uniformly on* K *by means of sums of tensor products* $p_1\otimes p_2$, p_j *positive polynomial on the positive semiaxes, in variable* $t_j, j=1,2.$

Proof. If K is the support of a nonnegative function $f\in C_c(R^2_+)$, then

$$K \subset K_1 \times K_2, K_j = pr_j(K), \ j = 1,2.$$

Consider a rectangle R_2 containing the above Cartesian product of compacts and apply continuous approximation of the extension of f vanishing outside its support, by means of Luzin's theorem. Then approximate this continuous function on the rectangle by the corresponding Bernstein polynomials in two variables. Each term of such a polynomial is a tensor product $p_1 \otimes p_2$, of positive polynomials in each variable. Extend each p_j such that it vanishes outside $pr_j(R_2)$, applying then Luzin's theorem once more, $j = 1,2.$ This procedure does not change the values of p_j on $K_j, j = 1,2.$ One obtains approximation by sums of tensor products of positive continuous functions with compact support, in each variable $t_j, j = 1,2.$ The approximating process holds in L^1 norm on R_+^2 and uniformly on K.

Now application of lemma 2.2 leads to approximation of each such function in each separate variable by a dominating (positive) polynomial, in the space $L^1_{\nu_j}(R_+)$ and uniformly on $K_j, \ j = 1,2.$ The conclusion follows. □

3. Main results

Using approximation results mentioned above, the following statement holds true. The statement remains valid for measures with non-compact support. In the sequel Y will be an order complete Banach lattice with solid norm: $|y_1| \le |y_2| \Rightarrow \|y_1\| \le \|y_2\|.$ Let

$$X = C(K), \phi_{j,k} \in X, \phi_{j,k}(t_1,t_2) = t_1^j t_2^k, (j,k) \in \mathrm{N}^2, G \in B(X,Y)$$

a given linear positive (bounded) operator.

Theorem 3.1. *Let* $K_1 \subset R_+, K_2 \subset R_+$ *be compact subsets and* $K = K_1 \times K_2.$

Let $(y_{j,k})_{(j,k)\in N^2}$ *be a sequence in* $Y.$ *The following statements are equivalent*

(a) *there exists a unique operator* F *satisfying the conditions*

$$F \in B(X,Y),\ F(\phi_{j,k}) = y_{j,k},\ \forall (j,k) \in N^2,$$
$$0 \le F(\psi) \le G(\psi),\ \forall \psi \in X_+, \|F\| \le \|G\|;$$

(b) *for any finite subsets* $J_1, J_2 \subset N,$ *and any* $\{\alpha_j\}_{j\in J_1}, \{\beta_k\}_{k\in J_2},$ *we have:*

$$0 \le \sum_{\substack{i,j\in J_1,\\ k,l\in J_2}} \alpha_i\alpha_j\beta_k\beta_l y_{(i+j,k+l)} \le \sum_{\substack{i,j\in J_1,\\ k,l\in J_2}} \alpha_i\alpha_j\beta_k\beta_l \cdot G(\phi_{i+j,k+l});$$
$$0 \le \sum_{\substack{i,j\in J_1,\\ k,l\in J_2}} \alpha_i\alpha_j\beta_k\beta_l y_{(i+j+1,k+l)} \le \sum_{\substack{i,j\in J_1,\\ k,l\in J_2}} \alpha_i\alpha_j\beta_k\beta_l \cdot G(\phi_{i+j+1,k+l})$$
$$0 \le \sum_{\substack{i,j\in J_1,\\ k,l\in J_2}} \alpha_i\alpha_j\beta_k\beta_l y_{(i+j,k+l+1)} \le \sum_{\substack{i,j\in J_1,\\ k,l\in J_2}} \alpha_i\alpha_j\beta_k\beta_l \cdot G(\phi_{i+j,k+l+1});$$
$$0 \le \sum_{\substack{i,j\in J_1,\\ k,l\in J_2}} \alpha_i\alpha_j\beta_k\beta_l y_{(i+j+1,k+l+1)} \le \sum_{\substack{i,j\in J_1,\\ k,l\in J_2}} \alpha_i\alpha_j\beta_k\beta_l \cdot G(\phi_{i+j+1,k+l+1})$$

Proof. Following Lemma 2.3, one approximates uniformly on K any nonnegative function from $C(K)$ by sums of tensor products of positive polynomials on the positive semiaxis, in each separate variable $t_j,\ j = 1,2.$ On the other hand, thanks to the form of such polynomials [1], the assertion (b) says that

$$0 \le F_0(p_1 \otimes p_2) \le G(p_1 \otimes p_2),\ p_j(t_j) \ge 0,\ \ \forall t_j \ge 0,\ j = 1,2,$$

where F_0 is defined on the subspace of polynomials, such that the moment conditions are accomplished. Application of Theorem from Section 5.1.2 [20], p. 160, leads to the existence of a positive linear (hence also bounded) extension $F \in B(X,Y)$

of F_0. Now, using the density of the tensor products of positive polynomials in X_+, one obtains

$$0 \leq F(\psi) \leq G(\psi), \psi \in C(K), \psi \in X, \psi \geq 0.$$

To this end, we proceed in the following way. Let ψ be a continuous function on K. By the preceding Lemma 2.3, one approximates ψ on a rectangle containing K by sums of tensor products of positive polynomials on R_+

$$\sum_{l=0}^{k(m)} p_{m,1.l} \otimes p_{m,2,l} \to \psi, m \to \infty,$$

uniformly on K. Moreover, using (b) and the continuity of F, one obtains:

$$F(\psi) = \lim_m F\left(\sum_{l=0}^{k(m)} p_{m,1,l} \otimes p_{m,2,l}\right) \leq \lim_m G\left(\sum_{l=0}^{k(m)} p_{m,1,l} \otimes p_{m,2,l}\right)$$
$$= G\left(\lim_m \left(\sum_{l=0}^{k(m)} p_{m,1,l} \otimes p_{m,2,l}\right)\right) = G(\psi), \quad \psi \in (C(K))_+ .$$

For an arbitrary function ϕ from $C(K)$, we have:

$$|F(\phi)| \leq F(\phi^+) + F(\phi^-) = F(|\phi|) \leq G(|\phi|), \phi \in C(K)$$
$$\Rightarrow \|F(\phi)\| \leq \|G\| \cdot \|\phi\|_\infty , \phi \in C(K) \Rightarrow \|F\| \leq \|G\|.$$

The conclusion is that F is a linear positive operator dominated by G on the positive cone of $C(K)$, and $\|F\| \leq \|G\|$. This proves $(b) \Rightarrow (a)$. Since the converse implication is obvious, the proof is complete. □

The following extension result for linear operators has a nice geometric meaning and leads to interesting results concerning the extension of linear functionals and operators.

If V is a convex neighborhood of the origin in a locally convex space, we denote by p_V the gauge attached to V.

Theorem 3.2. *Let X be a locally convex space, Y an order complete vector lattice with strong order unit u_0 and $S \subset X$ a vector subspace. Let $A \subset X$ be a convex subset with the following qualities:*

(a) *there exists a neighborhood V of the origin such that $(S+V)\bigcap A = \Phi$*

(*A and S are distanced*);

(b) *A is bounded.*

Then for any equicontinuous family of linear operators $\{f_j\}_{j\in J} \subset L(S,Y)$ and for any $\tilde{y} \in Y_+ \setminus \{0\}$, there exists an equicontinuous family $\{F_j\}_{j\in J} \subset L(X,Y)$ such that

$$F_j|_S = f_j \quad and \quad F_j|_A \geq \tilde{y}, \forall j \in J.$$

Moreover, if V is a neighborhood of the origin such that

$$f_j(V \cap S) \subset [-u_0, u_0], \quad (S+V)\bigcap A = \Phi,$$
$$0 < \alpha \in R \text{ s.t. } p_V|_A \leq \alpha, \quad \alpha_1 > 0 \text{ s.t. } \tilde{y} \leq \alpha_1 u_0,$$

then the following relations hold

$$F_j(x) \leq (1+\alpha+\alpha_1)\, p_V(x)\cdot u_0, \quad x \in X,\ j \in J.$$

In oder to apply the above general theorem to concrete spaces, let $X = \bar{H}_r$ be the space of all continuous functions in the polydisc $\bar{D} = \prod_{k=1}^{n}\{|z_k| \leq r_k\},\ r_k > 1,\ k = 1,\dots,n$ which can be writen as power series with real coefficients, centered at $(0,\dots,0)$ in the open polydisc D. The order relation on X is given by the positive cone of all power series with nonnegative coefficients. Let

$$\phi_j(z_1,...,z_n) = z_1^{j_1}\cdots z_n^{j_n},\ j=(j_k)_{k=1}^n,\ |j| = \sum_{k=1}^n j_k \ge 1$$

and $A_k,\ k=1,...,n$ linear positive selfadjoint commuting operators on H, such that $\|A_k\| < r_k,\ k=1,...,n.$ We denote:

$$Y_1 = \{U \in \mathrm{A}(H);\ A_k U = UA_k,\ k=1,...,n\},\ Y = \{V \in Y_1; VU = UV\ \forall U \in Y_1\}.$$

Then Y is an order complete Banach lattice and a commutative Banach algebra of selfadjoint operators [20], [7].

Theorem 3.3. *Let* $(B_j)_{j\in \mathrm{N}^n},\ |j|\ge 1$ *be sequence in* $Y,\ \varepsilon > 0$ *such that*

$$|B_j| \le A_1^{j_1}\cdots A_n^{j_n} + \varepsilon\cdot I,\ \forall j=(j_1,...,j_n)\in \mathrm{N}^n,\ |j|\ge 1.$$

Let $\tilde{B} \in Y_+,\ \{\psi_j\}_{j\in\mathrm{N}^n} \subset X,\ \psi_j(0,...,0)=1,\ \|\psi_j\|_\infty \le 1\ \forall j\in \mathrm{N}^n.$ *Then there is a linear bounded operator* $F\in B(X,Y)$ *such that*

$$F(\varphi_j) = B_j,\ |j|\ge 1,\ F(\psi_j) \ge \tilde{B},\ j\in \mathrm{N}^n,$$

$$|F(\varphi)| \le \left(2 + \|\tilde{B}\|\cdot\left(\prod_{k=1}^n \frac{r_k}{r_k - \|A_k\|} + \varepsilon \prod_{k=1}^n \frac{r_k}{r_k - 1}\right)^{-1}\right)\cdot \|\varphi\|_\infty \cdot u_0,$$

$$u_0 := \left(\prod_{k=1}^n \frac{r_k}{r_k - \|A_k\|} + \varepsilon \prod_{k=1}^n \frac{r_k}{r_k - 1}\right)\cdot I$$

Proof. We apply theorem 3.2 to $S = Sp\{\phi_j;\ j\in \mathrm{N}^n, |j|\ge 1\},\ A = co(\{\psi_j;\ j\in\mathrm{N}^n\}).$

Conditions imposed on the values at $(0,\cdots,0)$ and on the norms of the functions ψ_j

lead to

$$\left\|\phi_j-\psi_m\right\|\geq\left|\phi_j(0,\ldots,0)-\psi_m(0,\ldots,0)\right|=1\Rightarrow$$
$$\left(S+B(0,1)\right)\cap A=\Phi\Rightarrow p_V(\cdot)=\|\cdot\|\Rightarrow p_{V|A}\leq 1=\alpha.$$

On the other hand, for

$$s\in S\cap B(0,1),\ f(s)=f\left(\sum_{j\in J_0}\lambda_j\phi_j\right)=\sum_{j\in J_0}\lambda_j B_j,$$

Cauchy inequalities for s and the hypothesis on the operators $|B_j|$ yield

$$|f(s)|=\left|\sum_{j\in J_0}\lambda_j B_j\right|\leq\sum_{j\in J_0}|\lambda_j|\cdot|B_j|\leq\|s\|\cdot\left(\sum_{j\in N^n}\frac{A_1^{j_1}\cdots A_n^{j_n}}{r_1^{j_1}\cdots r_n^{j_n}}\cdot+\varepsilon\sum_{j\in N^n}\frac{1}{r_1^{j_1}\cdots r_n^{j_n}}\cdot I\right)$$
$$\leq\left(\sum_{j_1\in\mathrm{N}}\|A_1\|^{j_1}/r_1^{j_1}\right)\cdots\left(\sum_{j_n\in\mathrm{N}}\|A_n\|^{j_n}/r_n^{j_n}\right)\cdot I+\varepsilon\cdot\left(\sum_{j_1\in\mathrm{N}}\frac{1}{r_1^{j_1}}\right)\cdots\left(\sum_{j_n\in\mathrm{N}}\frac{1}{r_n^{j_n}}\right)\cdot I=$$
$$=\left(\prod_{k=1}^{n}\frac{r_k}{r_k-\|A_k\|}+\varepsilon\prod_{k=1}^{n}\frac{r_k}{r_k-1}\right)\cdot I=u_0\Rightarrow-u_0\leq f(s)\leq u_0,\quad\forall s\in S\cap B(0,1).$$

We also have

$$\tilde{B}\leq\left\|\tilde{B}\right\|\cdot I=\left(\prod_{k=1}^{n}\frac{r_k}{r_k-\|A_k\|}+\varepsilon\prod_{k=1}^{n}\frac{r_k}{r_k-1}\right)^{-1}\cdot\left\|\tilde{B}\right\|\cdot u_0.$$

Now all conditions from the hypothesis of theorem 3.2 are accomplished. The conclusion follows. □

We recall the following result [13] on the abstract Markov moment problem, as an extension with two constraints theorem for linear operators. See also the classical statement [18].

Theorem 3.4. *Let* X *be an ordered vector space,* Y *an order complete vector lattice,* $\{x_j\}_{j\in J}\subset X,\{y_j\}_{j\in J}\subset Y$ *given families and* $F_1,F_2\in L(X,Y)$ *two linear* *operators. The following statements are equivalent:*

(a) *there is a linear operator* $F \in L(X,Y)$ *such that*

$$F_1(x) \le F(x) \le F_2(x)\ \forall x \in X_+,\ F(x_j) = y_j\ \ \forall j \in J;$$

(b) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j\in J_0} \subset R,$ *we have:*

$$\left(\sum_{j\in J_0} \lambda_j x_j = \psi_2 - \psi_1, \psi_1, \psi_2 \in X_+\right) \Rightarrow \sum_{j\in J_0} \lambda_j y_j \le F_2(\psi_2) - F_1(\psi_1).$$

The next result is an application of the preceding one, also using some ideas from [8]. Let X be the space of all power series with real coefficients, which are uniformly convergent in the closed unit disc, continuous up to the boundary, ordered by the positive cone of all series with nonnegative coefficients. We denote by $\phi_j \in X$ the basic functions $\phi_j(z) = z^j,\ j \in \mathrm{N}.$ Let Y be an order-complete Banach lattice and a Banach algebra, with multiplication unit $\bar{e} > 0, \|\bar{e}\| = 1.$ An example of such an algebra of selfadjoint operators is that preceding Theorem 3.3.

Theorem 3.5. *If* $a \in Y_+, \|a\| \le 1, (y_j)_{j\in \mathrm{N}}$ *is a sequence in* $Y,$ *and* $\varepsilon > 0$ *is a real number, then the following assertions are equivalent*

(a) *there is a bounded linear operator* $F \in B(X,Y)$ *such that*

$$F(\phi_j) = y_j,\ j \in \mathrm{N},\ 0 \le F(\psi) \le \psi(a) + \varepsilon\psi(\bar{e}),\ \forall \psi \in X_+,\ \|F\| \le 1 + \varepsilon;$$

(b) *we have*

$$0 \le y_j \le a^j + \varepsilon \cdot \bar{e},\ j \in \mathrm{N}.$$

Proof. The implication $(a) \Rightarrow (b)$ is obvious, since $\phi_j \in X_+$ and

$$y_j = F(\phi_j) \in \left[0, \phi_j(a) + \varepsilon\phi_j(\bar{e})\right] = \left[\ 0, a^j + \varepsilon \cdot \bar{e}\ \right].$$

To prove the converse, one applies Theorem 3.4 $(b) \Rightarrow (a).$ We verify the conditions of (b), Theorem 3.4:

$$\sum_{j\in J_0}\lambda_j\phi_j=\psi_2-\psi_1=\sum_{n\in\mathrm{N}}\alpha_n\phi_n-\sum_{n\in\mathrm{N}}\beta_n\phi_n,\ \alpha_n\geq 0,\beta_n\geq 0\Rightarrow$$
$$\lambda_j y_j\leq\lambda_j a^j+\varepsilon\lambda_j\overline{e}\leq\alpha_j a^j+\varepsilon\alpha_j\overline{e},\ \forall j\in J^+=\left\{j\in J_0;\ \lambda_j\geq 0\right\}.$$

These relations yield

$$\sum_{j\in J_0}\lambda_j y_j=\sum_{j\in J^+}\lambda_j y_j+\sum_{j\in J^-}\lambda_j y_j\leq$$
$$\sum_{j\in J^+}\lambda_j a^j+\varepsilon\cdot\left(\sum_{j\in J^+}\lambda_j\right)\cdot\overline{e}\leq\sum_{n\in\mathrm{N}}\alpha_n a^n+\varepsilon\left(\sum_{n\in\mathrm{N}}\alpha_n\right)\cdot\overline{e}=$$
$$\psi_2(a)+\varepsilon\cdot\psi_2(\overline{e})=F_2(\psi_2)-F_1(\psi_1),\ F_1\equiv 0.$$

Theorem 3.4 leads to the existence of a linear operator F, such that

$$F(\phi_j)=y_j,\ j\in\mathrm{N},\ F_1(\psi)=0\leq F(\psi)\leq F_2(\psi)=\psi(a)+\varepsilon\cdot\psi(\overline{e}),\ \psi\in X_+\Rightarrow$$
$$|F(\psi)|\leq|\psi|(a)+\varepsilon\cdot|\psi|(\overline{e}),\ \psi\in X\Rightarrow\|F(\psi)\|\leq\|\psi\|+\varepsilon\cdot\|\psi\|,\ \psi\in X\Rightarrow\|F\|\leq 1+\varepsilon.$$

This concludes the proof. □

Conflict of Interests

The author declares that there is no conflict of interests.

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