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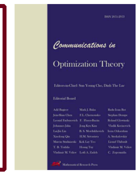

ITERATIVE PROCESSES FOR FIXED POINTS OF NONEXPANSIVE MAPPINGS

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Abstract. In this article, we investigate an iterative process to have strong convergence for common fixed points of an infinite family nonexpansive mappings. Convergence theorems are established in a real Banach space.

Keywords. Banach space; nonexpansive mapping; fixed point; iterative process.

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1. Introduction-Preliminaries

Let E be a real Banach space and let E^* be the dual space of E. Let

$$S(E) = \{x \in E : \|x\| = 1\}.$$

Then the norm of E is said to be Gâteaux differentiable if

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in S(E)$. In this case, E is said to be smooth. The norm of E is said to be uniformly Gâteaux differentiable if, for each $y \in S(E)$, the above limit is attained uniformly for $x \in S(E)$. The norm of E is said to be Fréchet differentiable if, for each $x \in S(E)$, the above limit is attained uniformly for $y \in S(E)$.

It is well known that (uniform) Fréchet differentiability of the norm of E implies (uniform) Gâteaux differentiability of the norm of E.

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Recall that the normalized duality mapping J from E into 2^{E^*} is defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2, \quad \|f\| = \|x\|\}, \quad \forall x \in E,$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

Recall that, if K and D are nonempty subsets of a Banach space E such that K is nonempty closed convex and $D \subset K$, then a mapping $Q : K \to D$ is sunny provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in K$ and $t \geq 0$ whenever $x + t(x - Q(x)) \in K$. A sunny nonexpansive retraction is a sunny retraction, which is also a nonexpansive mapping.

Let C be a nonempty closed and convex subset of E and $T : C \to C$ is a nonlinear mapping. In this paper, we use $F(T)$ to denote the fixed point set of the mapping T.

Recall that T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Recall that a mapping $f : C \to C$ is a contraction if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in C.$$

For such a case, f is also said to be an α-contraction. In this paper, we use Π_C to denote the collection of all contractions on C. That is, $\Pi_C = \{f | f : C \to C \text{ a contraction}\}$.

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping [1]. More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = t f(x) + (1 - t) T x, \quad \forall x \in C, \tag{1.1}$$

where $f \in \Pi_C$. Then Banach's Contraction Principle guarantees that T_t has a unique fixed point x_t in C.

Xu [2] proved that, if E is a uniformly smooth Banach space, then $\{x_t\}$ converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from Π_C onto $F(T)$. Iterative methods are popular tools to approximate fixed points of nonlinear mappings. Recall that the normal Mann's iteration was introduced by Mann [3] in 1953. Recently, construction of fixed points for nonlinear mappings via the normal Mann's iteration has been extensively investigated by many authors. Ishikawa iteration was introduced by Ishikawa [4] in 1974. Ishikawa iteration generates a sequence $\{x_n\}$ in the following manner:

$$\begin{cases} x_0 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \quad \forall n \geq 0, \end{cases}$$

where x_0 is an initial value and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$.

Therefore, many authors try to modify processes Mann and Ishikawa to have strong convergence for nonexpansive mappings; see [5-9] and the references therein.

In this paper, we investigate an iterative process to have strong convergence for common fixed points of an infinite family nonexpansive mappings. Convergence theorems are established in a real Banach space.

Lemma 1.1. [2] *Let E be a uniformly smooth Banach space, C be a closed convex subset of E, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $f \in \Pi_C$. Then the sequence $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t$ converges strongly to a point in $F(T)$. If we define a mapping $Q : \Pi_C \to F(T)$ by $Q(f) := \lim_{t\to 0} x_t, \quad \forall f \in \Pi_C$. Then $Q(f)$ solves solves the following variational inequality:*

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad f \in \Pi_C, p \in F(T).$$

Lemma 1.2. [10] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.3. *In a Banach space E, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall x, y \in E,$$

where $j(x+y) \in J(x+y)$.

Lemma 1.4. [11] *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(b) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

2. Main results

Theorem 2.1. *Let* C *be a nonempty closed and convex subset of a uniformly smooth Banach space. Let* $T : C \to C$ *be a nonexpansive mapping and* $f : C \to C$ *an* α*-contraction. Assume that* $F(T) \neq \emptyset$*. Let* $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be real number sequences in* $(0,1)$*. Let* $\{x_n\}$ *be a sequence generated in the following manner:* $x_0 \in C$ *and*

$$\begin{cases} y_n = \beta_n T\big(\gamma_n T x_n + (1-\gamma_n)x_n\big) + (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \quad \forall n \geq 0. \end{cases}$$

Assume that the following restrictions are satisfied.

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(b) *there exist constants* $b, b' \in (0,1)$ *such that* $0 < b \leq \beta_n \leq b' < 1$, $\forall n \geq 0$;
(c) *there exits a constant* $a \in (0,b]$ *such that* $\gamma_n \leq \frac{b-a}{2-b}$, $\forall n \geq 0$*;*
(d) $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$.

Then the sequence $\{x_n\}$ *converges strongly to some point in* $\mathscr{F}$*.*

Proof. Since the fixed point set of T is nonempty, we fix $p \in F(T)$. It follows that

$$\begin{aligned} \|y_n - p\| &\leq \beta_n \|Tz_n - p\| + (1-\beta_n)\|x_n - p\| \\ &\leq \beta_n \|z_n - p\| + (1-\beta_n)\|x_n - p\| \\ &\leq \|x_n - p\|, \end{aligned}$$

which yields that

$$\begin{aligned}\|x_{n+1}-p\| &\le \alpha_n\|f(x_n)-p\|+(1-\alpha_n)\|y_n-p\|\\ &\le \alpha_n\|f(x_n)-f(p)\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)\|y_n-p\|\\ &\le [1-\alpha_n(1-\alpha)]\|x_n-p\|+\alpha_n\|f(p)-p\|.\end{aligned}$$

By simple inductions, we obtain that

$$\|x_n-p\| \le \max\{\frac{\|p-f(p)\|}{1-\alpha}, \|x_0-p\|\}, \quad \forall n\ge 0.$$

This shows that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. Next, we show that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.1}$$

Put $v_n=\frac{x_{n+1}-(1-\beta_n)x_n}{\beta_n}$. It follows that

$$x_{n+1}=\beta_n v_n+(1-\beta_n)x_n, \quad \forall n\ge 0. \tag{2.2}$$

Now, we compute $\|v_{n+1}-v_n\|$. Note that

$$\begin{aligned}v_{n+1}-v_n=&\frac{\alpha_{n+1}f(x_{n+1})+(1-\alpha_{n+1})y_{n+1}-(1-\beta_{n+1})x_{n+1}}{\beta_{n+1}}\\ &-\frac{\alpha_n f(x_n)+(1-\alpha_n)y_n-(1-\beta_n)x_n}{\beta_n}.\end{aligned}$$

It follows that

$$\begin{aligned}\|v_{n+1}-v_n\| \le& \frac{\alpha_{n+1}}{\beta_{n+1}}\|f(x_{n+1})-y_{n+1}\|+\frac{\alpha_n}{\beta_n}\|y_n-f(x_n)\|\\ &+\|z_{n+1}-z_n\|.\end{aligned} \tag{2.3}$$

It follows that

$$\limsup_{n\to\infty}\left(\|v_{n+1}-v_n-\|x_{n+1}-x_n\|\right)\le 0.$$

In view of Lemma 1.2, we see that $\lim_{n\to\infty}\|v_n-x_n\|=0$, which combines with (2.2) shows that (2.1) holds. Now, we are in a position to show that $\lim_{n\to\infty}\|Tx_n-x_n\|=0$.

Note that

$$\begin{aligned}\|x_n-Tx_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|+\|y_n-Tz_n\|+\|Tz_n-Tx_n\|\\ &\le \|x_n-x_{n+1}\|+\alpha_n\|f(x_n)-y_n\|+(1-\beta_n)\|x_n-Tx_n\|\\ &\quad +(2-\beta_n)\gamma_n\|Tx_n-x_n\|.\end{aligned}$$

From the condition (c), we arrive at $a\|x_n - Tx_n\| \leq \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - y_n\|$,
In view of the conditions (a), (b), we obtain that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Next, we show that $\limsup_{n\to\infty}\langle x^* - f(x^*), J(x^* - p)\rangle \leq 0$, where $x^* = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction $x \mapsto tf(x) + (1-t)Tx$. Then x_t solves the fixed point equation $x_t = tf(x_t) + (1-t)Wx_t$. Thus we have $\|x_t - x_n\| = \|(1-t)(Tx_t - x_n) + t(f(x_t) - x_n)\|$. On the other hand, for any $t \in (0,1)$, we see that

$$\begin{aligned}\|x_t - x_n\|^2 &= (1-t)\big(\langle Tx_t - Tx_n, J(x_t - x_n)\rangle + \langle Tx_n - x_n, J(x_t - x_n)\rangle\big)\\ &\quad + t\langle f(x_t) - x_t, J(x_t - x_n)\rangle + t\langle x_t - x_n, J(x_t - x_n)\rangle\\ &\leq (1-t)\big(\|x_t - x_n\|^2 + \|Tx_n - x_n\|\|x_t - x_n\|\big)\\ &\quad + t\langle f(x_t) - x_t, J(x_t - x_n)\rangle + t\|x_t - x_n\|^2\\ &\leq \|x_t - x_n\|^2 + \|Tx_n - x_n\|\|x_t - x_n\| + t\langle f(x_t) - x_t, J(x_t - x_n)\rangle.\end{aligned}$$

It follows that $\langle x_t - f(x_t), J(x_t - x_n)\rangle \leq \frac{1}{t}\|Tx_n - x_n\|\|x_t - x_n\|, \quad \forall t \in (0,1)$.
It follows that $\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \leq 0$.
Since the fact that J is strong to weak* uniformly continuous on bounded subsets of E, we see that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\forall t \in (0,\delta)$ the following inequality holds $\langle f(x^*) - x^*, J(x_n - x^*)\rangle \leq \langle x_t - f(x_t), J(x_t - x_n)\rangle + \varepsilon$. This implies that $\limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*)\rangle \leq \limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle + \varepsilon$. Since ε is arbitrary, we see that $\limsup_{n\to\infty}\langle f(x^*) - x^*, J(x_n - x^*)\rangle \leq 0$. In view of Lemma 1.3, we see that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\|f(x_n) - f(x^*)\|\|x_{n+1} - x^*\|\\ &\quad + 2\alpha_n\langle f(x^*) - x^*, J(x_{n+1} - x^*)\rangle\\ &\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + \alpha_n\alpha(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2)\\ &\quad + 2\alpha_n\langle f(x^*) - x^*, J(x_{n+1} - x^*)\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \frac{(1-\alpha_n)^2+\alpha_n\alpha}{1-\alpha_n\alpha}\|x_n-x^*\|^2+\frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(x^*)-x^*,J(x_{n+1}-x^*)\rangle\\ &\le [1-\frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}]\|x_n-x^*\|^2\\ &\quad+\frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}[\frac{1}{1-\alpha}\langle f(x^*)-x^*,J(x_{n+1}-x^*)\rangle+\frac{\alpha_n}{2(1-\alpha)}M],\end{aligned}$$

where M is an appropriate constant such that $M \ge \sup_{n\ge 1}\{\|x_n-x^*\|^2\}$. Put $j_n=\frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}$ and $t_n=\frac{1}{1-\alpha}\langle f(x^*)-x^*,J(x_{n+1}-x^*)\rangle+\frac{\alpha_n}{2(1-\alpha)}M_4$. It follows that $\|x_{n+1}-x^*\|^2\le(1-j_n)\|x_n-x^*\|+j_nt_n,\quad \forall n\ge 0$. It follows from the conditions (a), (b) that $\lim_{n\to\infty} j_n=0$, $\sum_{n=0}^{\infty} j_n=\infty$, $\limsup_{n\to\infty} t_n\le 0$. In view of Lemma 1.4, we obtain that $x_n\to x^*$ as $n\to\infty$. This completes the proof.

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