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HYBRID PROJECTION ALGORITHMS FOR ASYMPTOTICALLY QUASI-ϕ-NONEXPANSIVE MAPPINGS

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Abstract. In this paepr, we investigate hybrid projection algorithms for asymptotically quasi-ϕ-nonexpansive mapping in the framework of Banach spaces. Our results improve and extend the corresponding ones announced by many others.

Keywords: asymptotic behavior; hybrid hybrid algorithm; nonexpansive mapping; generalized projection.

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1. Introduction

Recently, hybrid projection algorithm have been studied as an effective and powerful tool for studying a wide class of real world problems which arise in economics, finance, image reconstruction, transportation, and network; see [1-22] and the references therein. In this paepr, we investigate hybrid projection algorithms for asymptotically quasi-ϕ-nonexpansive mapping in the framework of Banach spaces. Strong convergence theorems of fixed points are established.

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2. Preliminaries

Let E be a Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single-valued and if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E.

As we all know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [7] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Consider the functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \text{for } x, y \in E. \tag{2.1}$$

Observe that, in a Hilbert space H, (2.1) reduces to $\phi(x,y) = \|x-y\|^2$, $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x) \tag{2.2}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J [7]. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y,x) \le (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{2.3}$$

Remark 2.1. If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x,y) = 0$ then $x = y$. From (2.3), we have $\|x\| = \|y\|$. This implies $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [8] for more details.

Let C be a closed convex subset of E, and let T be a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T [9] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F(T)}$. A mapping T from C into itself is said to be relatively nonexpansive if $\widetilde{F(T)} = F(T)$ and $\phi(p,Tx) \le \phi(p,x)$ for all $x \in C$ and $p \in F(T)$.

Let C be a closed convex subset of E, and T a mapping from C into itself. Recall that T is said to be ϕ-nonexpansive, if $\phi(Tx,Ty) \le \phi(x,y)$ for $x,y \in C$. T is said to be quasi-ϕ-nonexpansive if $F(T) \neq \emptyset$ and $\phi(p,Tx) \le \phi(p,x)$ for $x \in C$ and $p \in F(T)$. Recall also that T is said to be ϕ-asymptotically nonexpansive, if there exists some real sequence $\{k_n\}$ with $k_n \ge 1$ and $k_n \to \infty$ as $n \to \infty$ such that $\phi(T^n x, T^n y) \le k_n \phi(x,y)$ for $x,y \in C$. T is said to be quasi-ϕ-asymptotically nonexpansive if $F(T) \neq \emptyset$ and $\phi(p,T^n x) \le k_n \phi(p,x)$ for $x \in C$ and $p \in F(T)$. $T : C \to C$ is said to be asymptotically regular on C if, for any bounded subset K of C, there holds

$$\limsup_{n\to\infty}\{\|T^{n+1}x - T^n x\| : x \in K\} = 0.$$

Remark 2.2. The class of quasi-ϕ-asymptotically nonexpansive mappings is more general than the class of relatively asymptotically nonexpansive mappings [10,11] which requires the strong restriction: $F(T) = \widetilde{F(T)}$.

Remark 2.3. A ϕ-asymptotically nonexpansive mapping with a nonempty fixed point set $F(T)$ is a quasi-ϕ-asymptotically nonexpansive mapping, but the converse may be not true.

Next, we give some examples [12] which are closed quasi-ϕ-asymptotically nonexpansive.

Example 2.4. Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ is a maximal monotone mapping such that its zero set $A^{-1}0$ is nonempty. Then, $J_r = (J + rA)^{-1}$ is a closed quasi-ϕ-asymptotically nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.

Example 2.5. Let Π_C be the generalized projection from a smooth, strictly convex, and reflexive Banach space E onto a nonempty closed convex subset C of E. Then, Π_C is a closed quasi-ϕ-asymptotically nonexpansive mapping from E onto C with $F(\Pi_C) = C$.

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x,y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| =$

$1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$$

exists for each $x,y\in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x,y\in E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

We need the following lemmas for the proof of our main results.

Lemma 2.1 [13] *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n,y_n)\to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n-y_n\to 0$.*

Lemma 2.2 [7] *Let C be a nonempty closed convex subset of a smooth Banach space E and $x\in E$. Then, $x_0=\Pi_C x$ if and only if*

$$\langle x_0-y, Jx-Jx_0\rangle\geq 0\quad \forall y\in C.$$

Lemma 2.3 [7] *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x\in E$. Then*

$$\phi(y,\Pi_C x)+\phi(\Pi_C x,x)\leq\phi(y,x)\quad \forall y\in C.$$

Lemma 2.4. [12] *Let E be a uniformly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a closed quasi-ϕ-nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

3. Main results

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space E and let C be a nonempty and closed convex subset of E. Let $T:C\to C$ be a closed asymptotically quasi-ϕ-nonexpansive mapping such that $F(T)\neq\emptyset$ and $F(T)$ is bounded. Assume that T is* *asymptotically regular on C. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}\Big(\alpha_n Jx_n + (1-\alpha_n)JT^n\big(\Pi_C J^{-1}(\beta_n Jx_n + (1-\beta_n)JT^n x_n)\big)\Big), \\ C_{n+1} = \{u \in C_n : \phi(u,y_n) \le \phi(u,x_n) \\ \qquad +(1-\alpha_n)[h_n\|\beta_n Jx_n + (1-\beta_n)JT^n x_n)\|^2 - \|x_n\|^2 + (h_n-1)M \\ \qquad -2h_n\langle u, J(\beta_n Jx_n + (1-\beta_n)JT^n x_n))\rangle + 2\langle u, Jx_n\rangle]\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{3.1}$$

where M is an appropriate constant such that $M \ge \|w\|^2$ for any $w \in F(T)$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in [0,1] such that $\limsup_{n\to\infty} \alpha_n < 1$ *and* $\lim_{n\to\infty} \beta_n = 1$. *Then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$.*

Proof. Put $z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JT^n x_n)$. We show that C_n is closed and convex for all $n \ge 0$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$, where $\mathbb{N}$ denotes the natural numbers set. For $u \in C_k$, one obtains that

$$\phi(u,y_k) \le \phi(u,x_k) + (1-\alpha_k)[h_k\|z_k\|^2 - \|x_k\|^2 + (h_k-1)M - 2h_k\langle u, Jz_k\rangle + 2\langle u, Jx_k\rangle]$$

is equivalent to

$$\begin{aligned} &(1-\alpha_k)[2h_k\langle u, Jz_k\rangle - 2\langle u, Jx_k\rangle] + 2\langle u, Jx_k\rangle - 2\langle u, Jy_k\rangle \\ &\le \|x_k\|^2 - \|y_k\|^2 + (1-\alpha_k)[h_k\|z_k\|^2 - \|x_k\|^2 + (h_k-1)M]. \end{aligned}$$

It is easy to see that C_{k+1} is closed and convex. Then, for all $n \ge 0$, C_n is closed and convex. This shows that $\Pi_{C_{n+1}} x_0$ is well defined. Next, we prove $F(T) \subset C_n$ for all $n \ge 0$. $F(T) \subset C_1 = C$ is obvious. Suppose $F(T) \subset C_k$ for some $k \in \mathbb{N}$. Then, for $\forall w \in F(T) \subset C_k$, one has

$$\begin{aligned}
&\phi(w,y_k)\\
&\leq \|w\|^2 - 2\alpha_k\langle w, Jx_k\rangle - 2(1-\alpha_k)\langle w, JT^k z_k\rangle + \alpha_k\|x_k\|^2 + (1-\alpha_k)\|T^k z_k\|^2\\
&= \alpha_k\phi(w,x_k) + (1-\alpha_k)\phi(w,T^k z_k)\\
&\leq \alpha_k\phi(w,x_k) + (1-\alpha_k)h_k\phi(w,z_k)\\
&= \phi(w,x_k) + (1-\alpha_k)[h_k\phi(w,z_k) - \phi(w,x_k)]\\
&\leq \phi(w,x_k) + (1-\alpha_k)[h_k\|z_k\|^2 - \|x_k\|^2 + (h_k-1)\|w\|^2 - 2h_k\langle w, Jz_k\rangle + 2\langle w, Jx_k\rangle]\\
&\leq \phi(w,x_k) + (1-\alpha_k)[h_k\|z_k\|^2 - \|x_k\|^2 + (h_k-1)M - 2h_k\langle w, Jz_k\rangle + 2\langle w, Jx_k\rangle].
\end{aligned}$$

which shows $w \in C_{k+1}$. This implies that $F(T) \subset C_n$ for all $n \geq 0$. From $x_n = \Pi_{C_n} x_0$, one sees

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall z \in C_n. \tag{3.2}$$

Since $F(T) \subset C_n$ for all $n \geq 0$, we arrive at

$$\langle x_n - w, Jx_0 - Jx_n\rangle \geq 0, \quad \forall w \in F(T). \tag{3.3}$$

In view of Lemma 2.3, one obtains that

$$\phi(x_n,x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(w,x_0) - \phi(w,x_n) \leq \phi(w,x_0),$$

for each $w \in F(T) \subset C_n$ and for all $n \geq 0$. Therefore, the sequence $\phi(x_n,x_0)$ is bounded. On the other hand, noticing that $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, one has $\phi(x_n,x_0) \leq \phi(x_{n+1},x_0)$ for all $n \geq 0$. Therefore, $\{\phi(x_n,x_0)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n,x_0)\}$ exists. By the construction of C_n, one has that $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for any positive integer $m \geq n$. It follows that

$$\phi(x_m,x_n) \leq \phi(x_m,x_0) - \phi(x_n,x_0). \tag{3.4}$$

Letting $m,n \to \infty$ in (3.4), one has $\phi(x_m,x_n) \to 0$. It follows from Lemma 2.1 that $x_m - x_n \to 0$ as $m,n \to \infty$ Hence, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, one can assume that $x_n \to p \in C$ as $n \to \infty$.

Next, we show $p=\Pi_{F(T)}x_0$. To show this, we first show $p\in F(T)$. By taking $m=1$ in (3.4), one arrives at

$$\lim_{n\to\infty}\phi(x_{n+1},x_n)=0. \tag{3.5}$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.6}$$

Noticing that $x_{n+1}\in C_{n+1}$, we obtain

$$\begin{aligned}\phi(x_{n+1},y_n)\le \phi(x_{n+1},x_n)+(1-\alpha_n)[h_n\|z_n\|^2-\|x_n\|^2\\ +(h_n-1)M-2h_n\langle x_{n+1},Jz_n\rangle+2\langle x_{n+1},Jx_n\rangle].\end{aligned}$$

Next, we show

$$\lim_{n\to\infty}\phi(x_{n+1},y_n)=0.$$

Notice that

$$\phi(x_{n+1},z_n)\le\beta_n\phi(x_{n+1},x_n)+(1-\beta_n)\phi(x_{n+1},T^nx_n).$$

Since $\lim_{n\to\infty}\beta_n=1$ and (3.5), one has has

$$\phi(x_{n+1},z_n)\to 0,\quad as\ n\to\infty. \tag{3.7}$$

Notice that

$$\begin{aligned}&h_n\|z_n\|^2-\|x_n\|^2-2h_n\langle x_{n+1},Jz_n\rangle+2\langle x_{n+1},Jx_n\rangle\\ &=h_n(\|z_n\|^2+\|x_{n+1}\|^2-2\langle x_{n+1},Jz_n\rangle)+2\langle x_{n+1},Jx_n\rangle-h_n\|x_{n+1}\|^2-\|x_n\|^2\\ &\le h_n\phi(x_{n+1},z_n)+2\|x_{n+1}\|\|x_n\|-\|x_{n+1}\|^2-\|x_n\|^2\\ &\le h_n\phi(x_{n+1},z_n).\end{aligned} \tag{3.8}$$

It follows from (3.5), (3.7) and (3.8) that

$$\lim_{n\to\infty}\phi(x_{n+1},y_n)=0. \tag{3.9}$$

Using Lemma 2.1, we obtain

$$\lim_{n\to\infty}\|x_{n+1}-y_n\|=0. \tag{3.10}$$

Since J is uniformly norm-to-norm continuous on bounded sets, one has

$$\lim_{n\to\infty}\|Jx_{n+1}-Jy_n\|=\lim_{n\to\infty}\|Jx_{n+1}-Jx_n\|=0. \tag{3.11}$$

Noticing that

$$\|Jx_{n+1} - JT^n z_n\| \leq \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \alpha_n\|Jx_n - Jx_{n+1}\|).$$

From (3.11) and $\limsup_{n\to\infty} \alpha_n < 1$, we obtain

$$\lim_{n\to\infty} \|Jx_{n+1} - JT^n z_n\| = 0.$$

Therefore, we find that

$$\lim_{n\to\infty} \|x_{n+1} - T^n z_n\| = 0. \tag{3.12}$$

Hence one arrives at

$$\|x_n - T^n x_n\| \leq \|x_{n+1} - x_n\| + \|x_{n+1} - T^n z_n\|.$$

It follows from (3.6) and (3.12) that $\lim_{n\to\infty} \|T^n x_n - x_n\| = 0$. Noticing that $x_n \to p$ as $n \to \infty$, one has

$$T^n x_n \to p, \quad \text{as } n \to \infty. \tag{3.13}$$

On the other hand, one has

$$\|T^{n+1}x_n - p\| \leq \|T^{n+1}x_n - T^n x_n\| + \|T^n x_n - p\|.$$

From (3.13) and the asymptotic regularity of T, one obtains $T^{n+1}x_n \to p$; that is, $TT^n x_n \to p$. From the closedness of T, one gets $p = Tp$.

Finally, we show that $p = \Pi_{F(T)}x_0$. From $x_n = \Pi_{C_n}x_0$, one has

$$\langle x_n - w, Jx_0 - Jx_n \rangle \geq 0, \quad \forall w \in F(T) \subset C_n. \tag{3.14}$$

Taking the limit as $n \to \infty$ in (3.14), we obtain that

$$\langle p - w, Jx_0 - Jp \rangle \geq 0, \quad \forall w \in F(T),$$

and hence $p = \Pi_{F(T)}x_0$ by Lemma 2.2. This completes the proof.

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