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$\Delta -$ CONVERGENCE THEOREMS FOR MANN AND ISHIKAWA ITERATION PROCEDURES WITH ERRORS IN CAT(0) SPACES

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Abstract: In this paper we give the Δ-convergence results ofMann and Ishikawa iteration procedures with errors in CAT(0) spaces by using the concept of Δ-convergence in CAT(0) spaces introduced by Dhompongsa, Panyanak [On Δ-convergence theorems in CAT(0) spaces, Comput. Math. Appl., 56,(2008), 2572-2579.].

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1. Introduction:

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t),c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by [x, y]. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the

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geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) = \Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom: Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$, $d(x, y) \le d_{\mathbb{E}^2}(\bar{x}, \bar{y})$.

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \le \frac{1}{2} d(x, y_1)^2 + \frac{1}{2} d(x, y_2)^2 - \frac{1}{4} d(y_1, y_2)^2 \qquad \text{(CN)}$$

This is the (CN) inequality of Bruhat and Tits [5]. In fact, a geodesic space is a CAT(0) space if and only if it satisfy (CN) inequality.

Definition1.1 [8]: Let C be a non-empty subset of a CAT(0) space X and T: C $\rightarrow$ X be a mapping. T is called nonexpansive if for each $x, y \in C$,

$$d(Tx, Ty) \le d(x, y).$$

A point $x \in C$ is called a fixed point of T if Tx= x.We denote with F(T) the set of fixed points of T.

Definition 1.2 [3]: A mapping T of a metric space (X, d) into itself is said to be asymptotically regularif for any $x \in X$, the sequence $\{d(T^{n+1}x, T^n x)\} \to 0$ as $n \to \infty$.

Definition 1.3 [8]: Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$.The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$ and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$is the set $A(\{x_n\}) = \{x \in X: r(x, \{x_n\}) = r(\{x_n\})\}$.

Remark 1.4: In a CAT(0) space, $A(\{x_n\})$ consists of exactly one point (see, e.g., [7], Proposition 7).

Definition1.5 [8]: A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim x_n = x$ and call x the Δ-limit of $\{x_n\}$.

In 1953, W.R. Mann defined the Mann iteration [18] as

x_{n+1}=t_n Tx_n + (1-t_n)x_n, n = 0, 1, 2,…..
where $\{t_n\}$ is a sequence in [0, 1].
In 1974, S. Ishikawa defined the Ishikawa iteration [11] as
x_{n+1}=t_nT (s_nT x_n + (1- s_n)x_n) + (1 $-t_n$)x_n, n =0, 1, 2, . . .
where $\{t_n\}$ and $\{s_n\}$ are sequences in [0, 1].

We now modify the above iteration processes in CAT(0) spaces so the iterations becomes:

Mann iteration:x_{n+1}=t_n T$x_n \oplus$ (1-t_n)x_n, n = 0, 1, 2,…..where $\{t_n\}$ is a sequence in [0, 1].

Ishikawa iteration:x_{n+1}=t_nT (s_nT $x_n \oplus$ (1- s_n)x_n) $\oplus$ (1 $-t_n$)x_n,n =0, 1, 2, . . where $\{t_n\}$ and $\{s_n\}$ are sequences in [0, 1].

Mann iteration procedures with errors:

x_{n+1}=t_n T$x_n \oplus$ (1-t_n)$x_n \oplus u_n$,n= 0,1,2,….. (1.1)

Ishikawa iteration procedures with errors:

x_{n+1}=t_nT (s_nT $x_n \oplus$ (1- s_n)$x_n \oplus u_n$) $\oplus$ (1 $-t_n$)$x_n \oplus v_n$

or in two steps it can be written as

y_n= s_nT $x_n \oplus$ (1- s_n)$x_n \oplus u_n$,

x_{n+1}=t_nT $y_n \oplus$ (1-t_n)$x_n \oplus v_n$, n = 0, 1, 2,….. (1.2)

Now we collect some results which are used in our main results:

Lemma 1.6 [8]: Let (X, d) be a CAT(0) space. Then

(i) (X, d) is uniquely geodesic.

(ii) Let p, x, y be points of X andα $\in$ [0, 1]. Let m_1 and m_2 denote, respectively, the points of [p, x] and [p, y] satisfying d(p,m_1) = α d(p, x)andd(p, m_2) = α d(p, y).Then d(m_1, m_2) $\leq$ α d(x, y). (1.3)

(iii) Let x, y ∈X, x ≠ y and z, w ∈ [x, y] such that d(x, z) = d(x, w). Then z = w.

(iv) Let x, y ∈ X. For each t ∈ [0, 1], there exists a unique point z∈ [x, y] such that d(x, z) = t d(x, y) and d(y, z) = (1 − t) d(x, y). (1.4)

Lemma 1.7 [8]: Let X be a CAT(0) space and let x, y ∈ X such that x≠ y. Then

(i) [x, y] = {(1 − t) x $\oplus$ t y : t ∈ [0, 1]}.

(ii) d(x, z) + d(z, y) = d(x, y) if and only if z ∈ [x, y].

(iii) The mapping f: [0, 1] → [x, y], f (t) = (1 − t) x $\oplus$ty is continuous and bijective.

Lemma 1.8 [8]: (i) Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.

(ii) If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C then the asymptotic center of $\{x_n\}$ is in C.

(iii) If C is a closed convex subset of a complete CAT(0) space and if T : C→ X is a nonexpansive mapping then the conditions, $\{x_n\}$Δ-converges to xandd(x_n, T (x_n)) → 0 imply x∈ C and T(x) = x.

Lemma 1.9[8]: If $\{x_n\}$ is a bounded sequence in X with A($\{x_n\}$) = {x} and $\{u_n\}$ is a subsequence of $\{x_n\}$ with A($\{u_n\}$) = {u} andthe sequence {d(x_n, u)} converges then x = u.

Lemma 1.10 [8]: Let C be a closed convex subset of X and T: C → X be a nonexpansive mapping. Suppose $\{x_n\}$ is a bounded sequence in C such that $\lim_n$ d(x_n, Tx_n) =0 and {d(x_n, v)} converges for all v∈ F(T)then $\omega_w(x_n)$⊂F(T). Here$\omega_w(x_n)$= ∪A($\{u_n\}$) where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover,$\omega_w(x_n)$ consists of exactly onepoint.

Lemma 1.11 [8]: For each p ∈ F (T), $\lim_n d(x_n, p)$ exists.

Lemma 1.12 [8]:([10, Theorem 1]). Let (M, d) be a space of hyperbolic type. Let x_0 ∈ M and b < 1 and also let $\{\alpha_n\}$ ⊂[0, b] satisfy $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose T: M→ M is non-expansive and for each n, suppose x_{n+1} is the point of [x_n, Tx_n] which satisfies d(x_n, x_{n+1}) = α_nd(x_n, Tx_n), n = 0,1,2,……… If $\{x_n\}$ is bounded then $\lim_{n\to\infty}$ d(x_n, Tx_n) = 0.

In 2000, Deng Lei and Li Shenghong [16] proved the following lemma:

Lemma 1.13: Suppose that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are three sequences of nonnegative numbers such that $a_{n+1} \leq (1+b_n)a_n + c_n$ for all n ≥1. If $\sum_n b_n$and $\sum_n c_n$converges then $\lim_n a_n$ exists.

Theorem 1.14 [8]: Let C be a bounded closed convex subset of X and let T: C→ C be a nonexpansive asymptotically regular mapping. Then for any x_0 ∈C, the Picard iterate sequence { $T^n x_0$} is Δ-convergent to an element of F(T).

2. Main Results:

Lemma 2.1: Let X be a CAT(0) space. Then

$$d((1-t)x \oplus t y \oplus u, z) \leq (1-t)\, d(x, z) + t\, d(y, z) + d(u, z) \qquad (2.1)$$

for all x , y, z , u ∈ X and t ∈ [0, 1].

Proof: Let x, y, z ∈ X and t ∈ [0, 1]. Suppose that d(z, y) ≤ d(z, x).

Let p = (1 − t) x ⊕ t y and letx_0 be the point of [z, x] such that d(z, x_0) = d(z, y).

Put q = $(1 - t)\, x_0 \oplus t\, y$ and r = $(1 - t)\, x_0 \oplus t\, z$.

By Lemma 1.6(ii) [8], $d(z, q) \leq d(z, r) + d(r, q)$

$\leq (1 - t)\, d(x_0, z) + t\, d(z, y)$

$= d(z, y)$.

Suppose that (2.1) does not hold. Then

$(1- t)\, d(x, z) + t\, d(y, z) + d(u, z) \leq d(z, p \oplus u)$

$\leq d(z, p) + d(z, u)$ (by lemma 1.8, [8])

or $(1- t)\, d(x, z) + t\, d(y, z) \leq d(z, p)$

$\leq d(z, q) + d(q, p)$

$= d(z, (1-t)\, x_0 \oplus t\, y) + d(q, p)$

$\leq (1-t)\, d(z, x_0) + t\, d(z, y) + d(q, p)$

$= (1-t)\, d(z, y) + t\, d(z, y) + d(q, p)$

$= d(z, y) + d(q, p)$

yielding $d(q, p) > (1-t)\, \{d(z, x) - d(z, y)\}$

$= (1-t)\, d(x, x_0)$

This contradicts (1.3).Hence the result.■

Lemma 2.2: Let (X, d) be a CAT(0) space. Then

$$d((1 - t)\, x \oplus t\, y \oplus u, z)^2 \leq (1 - t)\, d(x, z)^2 + t\, d(y, z)^2 + d(u, z)^2 - t(1 - t)\, d(x, y)^2 - (1- t)\, d(x, u)^2 - t\, d(y, u)^2 \quad (2.2)$$

for all $t \in [0, 1]$ and $x, y, z, u \in X$.

Proof: We first prove the result for $t = \frac{k}{2^n}$, where $k, n \in \mathbb{N}$ are such that $k \leq 2^n$. We use induction on n. If n = 0, then $\frac{k}{2^n} = k$ and $k \in \{0, 1\}$.

If k = 0, then (2.2) $\Leftrightarrow d(1.x \oplus 0.y \oplus u, z)^2 \leq 1.d(x,z)^2 + 0 + d(u, z)^2 - 0 - 1.d(x, u)^2 - 0$

or $d(x \oplus u, z)^2 \leq d(x, z)^2 + d(u, z)^2 - d(x, u)^2$

or $d(x, z)^2 + d(u, z)^2 - d(x, u)^2 \leq d(x, z)^2 + d(u, z)^2 - d(x, u)^2$. (By using Lemma 1.9 [8])

If k = 1, then (2.2) $\Leftrightarrow d(0.x \oplus 1.y \oplus u, z)^2 \leq 0 + 1.\, d(y, z)^2 + d(u, z)^2 - 0 - 0 - 1.d(y, u)^2$.

or $d(y \oplus u, z)^2 \leq d(y, z)^2 + d(u, z)^2 - d(y, u)^2$

or $d(y, z)^2 + d(u, z)^2 - d(y, u)^2 \leq d(y, z)^2 + d(u, z)^2 - d(y, u)^2$.

Hence (2.2)is true for n = 0.

Now suppose that (2.2)is true for t = $\frac{k}{2^n}$. Hence

$$d\left(\left(1-\frac{k}{2^n}\right) x \oplus \frac{k}{2^n} y \oplus u , z\right)^2 \leq \left(1-\frac{k}{2^n}\right) d(x, z)^2 + \frac{k}{2^n} d(y, z)^2 + d(u , z)^2 - \left(1-\frac{k}{2^n}\right)\frac{k}{2^n} d(x ,y)^2 - \frac{k}{2^n} d(y, u)^2 - \left(1-\frac{k}{2^n}\right) d(x, u)^2 \quad (2.3)$$

for all k ∈ ℕ , k ≤ 2^n and all x, y, z, u ∈ X.

We have to prove (2.2)for t = $\frac{k}{2^{n+1}}$, where k ∈ ℕ , k ≤ 2^{n+1} .If we denote w = $\left(1-\frac{k}{2^{n+1}}\right) x \oplus \frac{k}{2^{n+1}} y$, then we have to prove

$$d(w \oplus u , z)^2 \leq \left(1-\frac{k}{2^{n+1}}\right) d(x, z)^2 + \frac{k}{2^{n+1}} d(y, z)^2 + d(u, z)^2 - \left(1-\frac{k}{2^{n+1}}\right)\frac{k}{2^{n+1}} d(x, y)^2 - \frac{k}{2^{n+1}} d(y, u)^2 - \left(1-\frac{k}{2^{n+1}}\right) d(x , u)^2 \quad (2.4)$$

First, we show (2.4) for k ≤ 2^n, that is, $\frac{k}{2^n}$ ∈ [0,1]. Let $\alpha=\left(1-\frac{k}{2^n}\right)x \oplus \frac{k}{2^n}y$ and $\beta = \frac{1}{2} x \oplus \frac{1}{2}\alpha$.

Then, $d(x, \beta) = \frac{1}{2} d(x , \alpha) = \frac{1}{2} d\left(x, \left(1-\frac{k}{2^n}\right)x \oplus \frac{k}{2^n}y\right) = \frac{k}{2^{n+1}} d(x, y) = d(x, w)$.Since α ∈ [x, y] and β ∈ [x ,α] then β ∈ [x , y].Since w ∈ [x, y] and d(x, β) = d(x, w), β = w, by Lemma 1.6 (iii) [8].

Now applying (CN) and the induction hypothesis, it follows that

$$d(w \oplus u , z)^2 = d\left(\frac{1}{2} x \oplus \frac{1}{2}\alpha \oplus u , z\right)^2$$

$$\leq \frac{1}{2} d(x, z)^2 + \frac{1}{2} d(\alpha, z)^2 + d(u, z)^2 - \frac{1}{4} d(x, \alpha)^2 - \frac{1}{2} d(x, u)^2 - \frac{1}{2} d(\alpha, u)^2$$

$$= \frac{1}{2} d(x, z)^2 + \frac{1}{2} d\left(\left(1-\frac{k}{2^n}\right) x \oplus \frac{k}{2^n}y, z\right)^2 + d(u, z)^2 - \frac{1}{4} d\left(x,\left(1-\frac{k}{2^n}\right)x \oplus \frac{k}{2^n}y \right)^2 - \frac{1}{2} d(x, u)^2 - \frac{1}{2} d\left(\left(1-\frac{k}{2^n}\right) x \oplus \frac{k}{2^n}y, u\right)^2$$

$$\leq \frac{1}{2} d(x, z)^2 + \frac{1}{2}\left(1-\frac{k}{2^n}\right) d(x, z)^2 + \frac{k}{2^n}\frac{1}{2} d(y, z)^2 - \frac{1}{2}\frac{k}{2^n}\left(1-\frac{k}{2^n}\right) d(x, y)^2 + d(u, z)^2 + 0 - \frac{1}{4}\frac{k}{2^n} d(x, y)^2 + \frac{1}{4}\frac{k}{2^n}\left(1-\frac{k}{2^n}\right) d(x, y)^2 - \frac{1}{2} d(x, u)^2 - \frac{1}{2}\left(1-\frac{k}{2^n}\right) d(x, u)^2 - \frac{1}{2}\frac{k}{2^n} d(y, u)^2 + \frac{1}{2}\frac{k}{2^n}\left(1-\frac{k}{2^n}\right) d(x, y)^2$$

$$= \frac{1}{2}\left(1+1-\frac{k}{2^n}\right) d(x, z)^2 + \frac{k}{2^{n+1}} d(y, z)^2 + d(x, y)^2 \left[-\frac{k}{2^{n+1}}\left(1-\frac{k}{2^n}\right) - \frac{1}{4}\frac{k}{2^n} - \frac{1}{4}\frac{k}{2^n}\left(1-\frac{k}{2^n}\right) + \frac{k}{2^{n+1}}\left(1-\frac{k}{2^n}\right)\right] + d(u, z)^2 - \frac{1}{2} d(x, u)^2 \left[1+1-\frac{k}{2^n}\right] - \frac{k}{2^{n+1}} d(y, u)^2.$$

$= \left(1-\frac{k}{2^{n+1}}\right) d(x, z)^2 + \frac{k}{2^{n+1}} d(y, z)^2 + d(u, z)^2 - \frac{k}{2^{n+1}}\left(1-\frac{k}{2^{n+1}}\right) d(x, y)^2 - \left(1-\frac{k}{2^{n+1}}\right) d(x, u)^2 - \frac{k}{2^{n+1}} d(y, u)^2.$

Now suppose that $2^n \le k \le 2^{n+1}$ and let p = 2^{n+1}- k. Then p $\le 2^n$, by applying (2.4) for p, we obtain

$d(w \oplus u , z)^2 = d(\frac{p}{2^{n+1}}x \oplus \left(1-\frac{p}{2^{n+1}}\right)y \oplus u, z)^2$

$\le \frac{p}{2^{n+1}} d(x, z)^2 + \left(1-\frac{p}{2^{n+1}}\right) d(y, z)^2 + d(u, z)^2 - \frac{p}{2^{n+1}}\left(1-\frac{p}{2^{n+1}}\right) d(x, y)^2 - \frac{p}{2^{n+1}} d(x, u)^2 - \left(1-\frac{p}{2^{n+1}}\right) d(y, u)^2.$

$= \left(1-\frac{k}{2^{n+1}}\right) d(x, z)^2 + \frac{k}{2^{n+1}} d(y, z)^2 + d(u, z)^2 - \frac{k}{2^{n+1}}\left(1-\frac{k}{2^{n+1}}\right) d(x, y)^2 - \left(1-\frac{k}{2^{n+1}}\right) d(x, u)^2 - \frac{k}{2^{n+1}} d(y, u)^2.$

In the following, we use the fact that the set D = $\{\frac{k}{2^n}: k, n \in \mathbb{N}, k \le 2^n\}$is dense in [0, 1]. Let t $\in$ [0,1]. Then there exists a sequence $\{t_k\}$ in D such that $\lim_{k\to\infty} t_k$= t. Now, we have

$d((1-t_k) x \oplus t_k y \oplus u , z)^2 \le (1-t_k) d(x, z)^2 + t_k d(y, z)^2 + d(u, z)^2 - (1-t_k) t_k d(x, y)^2 - t_k d(y, u)^2 - (1-t_k) d(x, u)^2.$

Letting k → ∞ and using Lemma 1.7 (iii) [8] and the fact that d is continuous, we get (2.2). ■

Lemma 2.3: Let C be a non-empty bounded closed convex subset of X and T: C → C be a nonexpansive mapping and $\{x_n\}$ be the sequence of Ishikawa iteration procedure with errors defined by (1.2)with sequences $\{t_n\}$ and $\{s_n\}$ in [0,1] such that $\sum_{n=0}^{\infty} t_n(1-t_n) = \infty$ and $\sum_{n=0}^{\infty} s_n(1-t_n) < \infty$ and $\limsup_n s_n < 1$. Then $\lim_{n\to\infty} d(Tx_n , x_n) = 0$.

Proof: For each n, let $y_n = s_n T x_n \oplus (1- s_n)x_n \oplus u_n$, thus

$x_{n+1} = t_n T y_n \oplus (1-t_n)x_n \oplus v_n$. Let p $\in$F(T). We set

$M = sup_{n\ge0}\{ d(Tx_{n+1}, v_n), d(x_n, u_n), d(y_n, v_n), d(x_n, v_n), d(T y_n, u_n) \}$.

Then M → 0 as n → ∞ and also $d(v_n, p)^2$, $d(u_n, p)^2 \to 0$ as n → ∞.

By Lemma 2.2 and the nonexpansiveness of T, we have

$d(x_{n+1} , p)^2 = d(t_n T y_n \oplus (1-t_n)x_n \oplus v_n , p)^2$

$\le t_n d(T y_n, p)^2 + (1-t_n) d(x_n, p)^2 + d(v_n, p)^2 - t_n(1-t_n) d(T y_n, x_n)^2$

$\le t_n d(y_n, p)^2 + (1-t_n) d(x_n, p)^2 - t_n(1-t_n) d(T y_n, x_n)^2$ (2.5)

and $d(y_n, p)^2 = d(s_n T x_n \oplus (1-s_n)x_n \oplus u_n , p)^2$

$\leq s_n$d(T x_n, p)^2 + (1-s_n) d(x_n, p)^2+ d(u_n, p)^2-s_n(1-s_n)d(T x_n,x_n)^2

$\leq s_n$d(x_n, p)^2 + (1-s_n) d(x_n, p)^2-s_n(1-s_n)d(T x_n,x_n)^2

= d (x_n, p)^2- s_n(1-s_n) d(T x_n,x_n)^2

$\leq$d (x_n, p)^2 (2.6)

From (2.5) and (2.6), we get

d(x_{n+1} , p)^2 $\leq t_n$d(x_n, p)^2 + (1-t_n) d(x_n, p)^2 - t_n(1-t_n)d(T y_n,x_n)^2

$\leq$ d (x_n, p)^2 - t_n(1-t_n)d(T y_n,x_n)^2 (2.7)

Thereforet_n(1-t_n) d(T y_n,x_n)^2 $\leq$d(x_n, p)^2 - d(x_{n+1} , p)^2

This implies $\sum_{n=1}^{\infty} t_n(1-t_n)$ d(T $y_n, x_n)^2$ $\leq$d(x_1, p)$^2 < \infty$

But, since $\sum_{n=1}^{\infty} t_n(1-t_n)$ diverges, we have $\lim_{n\to\infty} inf$ d(T y_n , x_n)^2 = 0 and thus $\lim_{n\to\infty} inf$ d(Ty_n, x_n) = 0. (2.8)

Since d(Tx_n,x_n) $\leq$d(Tx_n,Ty_n) + d(T y_n,x_n)

$\leq$d (x_n, y_n) + d(T y_n,x_n)

= d (x_n, s_nT $x_n \oplus$ (1- s_n)$x_n \oplus u_n$) + d(T y_n,x_n)

$\leq s_n$d(Tx_n,x_n) + 0 + d(x_n, u_n) + d(T y_n,x_n), using Lemma 2.1.

That is, d(Tx_n,x_n) $\leq \frac{1}{1-s_n}$ [M + d(T y_n,x_n)]

We have from (2.8) that $\lim_{n\to\infty} inf$ d(Tx_n, x_n) = 0 (2.9)

Since d(Tx_{n+1}, x_{n+1}) = d(Tx_{n+1}, t_nT $y_n \oplus$ (1-t_n)$x_n \oplus v_n$)

$\leq t_n$d(Tx_{n+1} , T y_n) + (1-t_n) d(Tx_{n+1}, x_n)+ d(Tx_{n+1} ,v_n)

$\leq t_n$d(x_{n+1} , y_n) + (1-t_n) [d(Tx_{n+1}, x_{n+1}) + d(Tx_{n+1}, x_n)] + M

= t_n d(t_nT $y_n \oplus$ (1-t_n)$x_n \oplus v_n$, y_n) + (1-t_n) [d(Tx_{n+1}, x_{n+1}) + d(t_nT $y_n \oplus$ (1-t_n)$x_n \oplus v_n$, x_n)] + M

= t_n [t_nd(T y_n, y_n) + (1- t_n) d(x_n , y_n) + M] + (1-t_n) [d(Tx_{n+1}, x_{n+1}) + t_nd(T y_n, x_n) + M] + M

= t_n [t_nd(Ty_n, y_n) + (1- t_n) d(x_n , y_n)] + (1-t_n) [d(Tx_{n+1}, x_{n+1}) + t_nd(Ty_n, x_n)]+ 2M

or d(Tx_{n+1}, x_{n+1}) $\leq t_n$d(T y_n, y_n) + (1- t_n)[d(x_n, y_n)+d(Ty_n, x_n)] + $\frac{2}{t_n}$M

= t_nd(Ty_n,s_nT$x_n \oplus$(1-s_n)$x_n \oplus u_n$)+(1-t_n)[d(x_n,y_n)+ d(Ty_n, x_n)] +$\frac{2}{t_n}$M

$\leq t_n s_n$ d(Ty_n, Tx_n) + (1-s_n) t_n d(Ty_n,x_n)+ t_nd(Ty_n, u_n) +(1- t_n) [d(x_n, y_n) + d(Ty_n, x_n)] + $\frac{2}{t_n}$M

$\leq t_n s_n$ d (y_n, x_n) + (1-s_n) t_n d(Ty_n,x_n) + (1- t_n) d(x_n, y_n) + (1- t_n) d(Ty_n, x_n) + ($t_n + \frac{2}{t_n}$)M

= (1- t_n+ $t_n s_n$) d(x_n, y_n) + (1- $t_n s_n$) d(Ty_n,x_n) + ($t_n + \frac{2}{t_n}$)M

= (1- t_n+$t_n s_n$) d(x_n, s_nT$x_n \oplus$(1-s_n)$x_n \oplus u_n$) + (1- $t_n s_n$) d(Ty_n,x_n) + ($t_n + \frac{2}{t_n}$)M

$\leq$(1- t_n+$t_n s_n$) [s_nd(x_n, Tx_n) + d(x_n, u_n)] + (1- $t_n s_n$) d(Ty_n,x_n) +($t_n + \frac{2}{t_n}$)M

$\leq s_n$(1- t_n+$t_n s_n$)d(x_n,T x_n) + (1- $t_n s_n$)[d(Ty_n,Tx_n)+d(Tx_n,x_n)]+(1+$t_n s_n$+$\frac{2}{t_n}$) M

$\leq s_n$(1- t_n+$t_n s_n$) d(x_n,Tx_n) +(1- $t_n s_n$) [d(y_n,x_n)+ d(Tx_n,x_n)]+(1+$t_n s_n$+$\frac{2}{t_n}$) M

= s_n(1- t_n+$t_n s_n$) d(x_n,Tx_n)+ (1- $t_n s_n$) [d(x_n ,s_nT$x_n \oplus$ (1-s_n)$x_n \oplus u_n$) + d(Tx_n,x_n)] + (1+$t_n s_n$+$\frac{2}{t_n}$) M

$\leq s_n$(1- t_n+$t_n s_n$) d(x_n,Tx_n) + (1- $t_n s_n$)[s_nd(Tx_n,x_n) + d(x_n, u_n)] + (1- $t_n s_n$) d(Tx_n,x_n) + (1+$t_n s_n$+$\frac{2}{t_n}$) M

= s_n(1- t_n+$t_n s_n$) d(x_n,Tx_n) + (1- $t_n s_n$) d(Tx_n,x_n) + (1- $t_n s_n$)s_nd(Tx_n,x_n) + (2+ $\frac{2}{t_n}$) M

= [1+ 2s_n(1- t_n)] d(x_n,Tx_n) + (2+ $\frac{2}{t_n}$) M

Since $\sum_n s_n(1 - t_n)$ converges and the sequence {d(x_n,Tx_n)} is bounded, it follows from Lemma 1.13 that $\lim_n$ d(x_n, Tx_n) exists and equals to zero by (2.9).■

Now we prove the Δ-convergence theorem of Mann iteration procedure with errors in CAT(0) spaces.

Theorem 2.4: Let C be a bounded closed convex subset of X and T: C → C a nonexpansive mapping. Then for any initial point $x_0$in C, the Mann iterative process with errors defined by (1.1), with the restrictions that$\sum_{n=0}^{\infty} t_n$diverges and lim$sup_n t_n$< 1, Δ-converges to a fixed point of T.

Proof: By Lemma 1.12, $\lim_n$ d(x_n, Tx_n) $=$ 0.As in the proof of Theorem 1.14, apply the fact that {d(x_n, v)} is convergent for each v∈ F(T) and Lemma 1.10 to conclude that{x_n} Δ-converges to an element of F(T).■

The next theorem is theΔ-convergence theorem of the Ishikawa iteration process with errors in CAT(0) spaces which is an analog of a result of Dhompongsa, Panyanak[8] and Tan and Xu [20].

Theorem 2.5: Let C be a bounded closed convex subset of X and T: C→ C a nonexpansive mapping. Then for any initial point x_0 in C, the Ishikawa iterate sequence $\{x_n\}$ defined by (1.2), with the restrictions that $\sum_{n=0}^{\infty} t_n(1-t_n)$ diverges and $\sum_{n=0}^{\infty} s_n(1-t_n)$ converges and $\limsup_n s_n < 1$, Δ-converges to a fixed point of T.

Proof: By Lemma 2.3,$\lim_n d(x_n, Tx_n) = 0$. As in the proof ofTheorem 1.14,apply the fact that $\{d(x_n, v)\}$ is convergent foreach v∈ F (T) and Lemma1.10to conclude that $\{x_n\}$ Δ-converges to an element of F (T).■

Now we give the strong convergence result of the Ishikawa iteration scheme with errors in a CAT(0) space setting.

Theorem 2.6: Let C be a bounded closed convex compact subset of X and T: C → C a nonexpansive mapping and $\{x_n\}$ is defined as (1.2).Then$\{x_n\}$converges strongly to a fixed point of T.

Proof: By the compactness of C, we see that $\{x_n\}$ has a strongly convergent subsequence $\{x_{n_k}\}$ whose limit we shall denoteby z. Then, by Lemma 2.3and the non-expansiveness of T,

$$d(z, Tz) \le d(z, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) + d(Tx_{n_k}, Tz)$$

$$\le 2\, d(z, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) \to 0 \text{ as } k \to \infty$$

Thereforez ∈F(T). By Lemma 1.11,$\lim_n d(x_n,z)$ exists.Thus z is the strong limit of the sequence $\{x_n\}$ itself.■

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