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TURAN INEQUALITIES FOR THE EXPONENTIAL INTEGRAL FUNCTIONS

W. T. SULAIMAN *

Department of Computer Engineering, College of Engineering, University of Mosul, Iraq

Abstract. In this paper, many new inequalities concerning the integral exponential functions are presented.

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1. Introduction

The exponential integral function denoted by $E_n(x)$ is defined by (see [1])

(1) $$E_n(x) = \int_1^\infty t^{-n} e^{-xt} dt, \quad x > 0, \quad n = 0, 1, 2, ...$$

The r-th derivative of $E_n(x), r$ is integer, denoted by $E_n^{(r)}(x)$ is easily given by

(2) $$E_n^{(r)}(x) = (-1)^r \int_1^\infty t^{r-n} e^{-xt} dt.$$

We define the incomplete exponential integral function by

(3) $$E_n(a, x) = \int_x^\infty t^{-n} e^{-at} dt, \quad x \geq 1, \quad a > 0, \quad n = 0, 1, 2, ...$$

*Corresponding author

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Clearly, $E_n(a,1) = E_n(a)$.

Laforgia and Natanini [2] proved the following inequality

$$E_n(x)E_m(x) \geq E_{\frac{n+m}{2}}(x). \tag{4}$$

The object of this paper is to present many inequalities concerning the exponential integral functions of the form

$$f(x+y) \leq (\geq) f(x) + f(y), \quad f(xy) \leq (\geq) f(x)f(y),$$

and others.

2. Results

Theorem 2.1. *For $a > 0$, let f be defined by $f(x) = E_n(x,a)/E_n(a)$. Then the following inequality holds.*

$$f(x+y+1) \geq f(x)f(y), \quad x, y \geq 1. \tag{5}$$

Proof. Define the function F by

$$F(x) = f(x)f(y) - f(x+y+1).$$

Therefore, on keeping y fixed, we have

$$\begin{aligned} F'(x) &= f'(x)f(y) - f'(x+y+1) \\ &= \frac{f(y)x^{-n}e^{-ax}}{E_n(a)} \left(\frac{e^{-ay}}{f(y)} \left(1 + \frac{y-1}{x}\right)^{-n} - 1 \right) \end{aligned}$$

Set $G(x) = \frac{e^{-ay}}{f(y)} \left(1 + \frac{y-1}{x}\right)^{-n} - 1$. Since

$$F(1) = f(y)f(1) - f(y) = 0, \quad \lim_{x\to\infty} F(x) = 0, \tag{6}$$

Then by Rolle's theorem there is a point $p \in (1,\infty)$ such that $F'(p) = 0$. $G(x)$ is increasing with respect to x with $G(p) = 0$. Therefore $G(x) < 0$ on $(0,p)$. As F is a positive multiple of $G(x)$, then $F'(x) < 0$, on $(0,p)$. Therefore $F(x)$ is decreasing on $(0,p)$, vanishing at p, and increasing on (p,∞). This fact together with (6) shows that $F(x)<0$, which implies (5). ☐

Theorem 2.2. *Let $x,y>0, m,n$ are integers, $p>1, \frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds*

$$E_{m+n}(\frac{x}{p}+\frac{y}{q}) \le E_{pm}^{1/p}(x)E_{qn}^{1/q}(y). \tag{7}$$

In particular

$$E_{m+n}^2(\frac{x+y}{2}) \le E_{2m}(x)E_{2n}(y). \tag{8}$$

Proof. We have

$$\begin{aligned} E_{m+n}(\frac{x}{p}+\frac{y}{q}) &= \int_1^\infty t^{-(m+n)}e^{-\left(\frac{x}{p}+\frac{y}{q}\right)t}dt \\ &= \int_1^\infty t^{-m}e^{-\frac{x}{p}t}t^{-n}e^{-\frac{y}{q}t}dt \\ &\le \left(\int_1^\infty t^{-pm}e^{-xt}dt\right)^{1/p}\left(\int_1^\infty t^{-qn}e^{-yt}dt\right)^{1/q} \\ &= E_{pm}^{1/p}(x)E_{qn}^{1/q}(y). \end{aligned}$$

(8) follows by putting $y=x$. ☐

Theorem 2.3. *Let $x,y>0$, then the following inequality holds.*

$$E_n(x+y) \le E_n(x)+E_n(y). \tag{9}$$

Proof. Set

$$\begin{aligned} f(x) &= E_n(x)+E_n(y)-E_n(x+y) \\ &= \int_1^\infty t^{-n}(e^{-xt}+e^{-yt}-e^{-(x+y)t})dt \end{aligned}$$

Then, on keeping y fixed we have

$$f'(x)=E_n'(x)-E_n'(x+y)=-\int_1^\infty t^{1-n}(e^{-xt}-e^{-(x+y)t})dt \le 0,$$

Therefore $f(x)$ is non-increasing. Since

$$\lim_{x\to\infty} f(x)=\int_1^\infty t^{-n}e^{-yt}dt \ge 0,$$

then $f(x) \geq 0$, which implies $E_n(x+y) \leq E_n(x) + E_n(y)$. □

Theorem 2.4. *Let* $x, y > 1, \frac{1}{x} + \frac{1}{y} \leq 1, p > 1, \frac{1}{p} + \frac{1}{q} = 1$. *Then the following inequality holds*

$$E_n(xy) \leq E_n^{1/p}(px) E_n^{1/q}(qy). \tag{10}$$

Proof. The hypothesis implies $x + y \leq xy$. Since

$$E_n'(x) = -\int_1^\infty t^{1-n} e^{-xt} dt \leq 0,$$

then $E_n(x)$ is non increasing. Therefore

$$\begin{aligned} E_n(xy) \leq E_n(x+y) &= \int_1^\infty t^{-n} e^{-(x+y)t} dt \\ &= \int_1^\infty t^{-\frac{n}{p}} e^{-xt} t^{-\frac{n}{q}} e^{-yt} dt \\ &\leq \left(\int_1^\infty t^{-n} e^{-pxt} dt \right)^{1/p} \left(\int_1^\infty t^{-n} e^{-qyt} dt \right)^{1/q} \\ &= E_n^{1/p}(px) E_n^{1/q}(qy). \end{aligned}$$

□

Theorem 2.5. *Let* $x, y > 0, y < 1, 0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$. *Then the following inequality holds*

$$E_n(xy) \geq E_n^{1/p}(px) E_n^{1/q}(qy). \tag{11}$$

Proof. By the hypothesis, we have $\frac{1}{x} + \frac{1}{y} \geq 1$ which implies $x + y \geq xy$. Since $E_n(x)$ is non increasing. Therefore

$$\begin{aligned} E_n(xy) \geq E_n(x+y) &= \int_1^\infty t^{-n} e^{-(x+y)t} dt \\ &= \int_1^\infty t^{-\frac{n}{p}} e^{-xt} t^{-\frac{n}{q}} e^{-yt} dt \\ &\geq \left(\int_1^\infty t^{-n} e^{-pxt} dt \right)^{1/p} \left(\int_1^\infty t^{-n} e^{-qyt} dt \right)^{1/q} \\ &= E_n^{1/p}(px) E_n^{1/q}(qy). \end{aligned}$$

□

Theorem 2.6. *Let* $x, y > 1, p > 1, 0 < r < 1, \frac{1}{p} + \frac{1}{q} = 1, \frac{1}{r} + \frac{1}{s} = 1$. *Then the following inequalities hold*

(a)

$$E_n(xy) \geq E_n^{1/r}\left(\frac{rx^p}{p}\right) E_n^{1/s}\left(\frac{sy^q}{q}\right).$$

(b)

$$E_n^2(xy) \leq E_n(x)E_n(y).$$

Proof. **(a)**

$$\begin{aligned} E_n(xy) &= \int_1^\infty t^{-n}e^{-xyt}dt \\ &\geq \int_1^\infty t^{-n}e^{-\left(\frac{x^p}{p}+\frac{y^q}{q}\right)}dt \\ &= \int_1^\infty t^{-\frac{n}{r}}e^{-\frac{x^p}{p}t}t^{-\frac{n}{s}}e^{-\frac{y^q}{q}t}dt \\ &\geq \left(\int_1^\infty t^{-n}e^{-\frac{rx^p}{p}t}\right)^{1/r}\left(\int_1^\infty t^{-n}e^{-\frac{sx^q}{q}t}\right)^{1/s} \\ &= E_n^{1/r}\left(\frac{rx^p}{p}\right)E_n^{1/s}\left(\frac{sy^q}{q}\right). \end{aligned}$$

(b). As $xy > x$, then

$$E_n(xy) = \int_1^\infty t^{-n}e^{-xyt}dt \leq \int_1^\infty t^{-n}e^{-xt}dt = E_n(x).$$

Also,

$$E_n(xy) \leq E_n(y).$$

Multiplying, we obtain (b). □

The coming inequalities concerning the multi-derivative of $E_n(x)$.

Theorem 2.7. *Let* $x, y > 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1, r > m + n, r$ *is an even integer.Then the following inequalities hold*

$$E_{m+n}^{(r)}\left(\frac{x}{p}+\frac{y}{q}\right) \leq (E_{pm}^{(r)}(x))^{1/p}(E_{qn}^{(r)}(y))^{1/q}, \tag{12}$$

$$E_n^{(r)}\left(\frac{x}{p}+\frac{y}{q}\right) \le (E_n^{(r)}(x))^{1/p}(E_n^{(r)}(y))^{1/q}, \tag{13}$$

$$\left(E_n^{(r)}\left(\frac{x+y}{2}\right)\right)^2 \le E_n^{(r)}(x)E_n^{(r)}(y). \tag{14}$$

Proof.

$$\begin{aligned} E_{m+n}^{(r)}\left(\frac{x}{p}+\frac{y}{q}\right) &= \int_1^\infty t^{r-m-n}e^{-\left(\frac{x}{p}+\frac{y}{q}\right)t}dt \\ &= \int_1^\infty t^{\frac{r}{p}-m}e^{-\frac{x}{p}t}t^{\frac{r}{q}-n}e^{-\frac{y}{q}t}dt \\ &\le \left(\int_1^\infty t^{r-pm}e^{-xt}dt\right)^{1/p}\left(\int_1^\infty t^{r-qn}e^{-yt}dt\right)^{1/q} \\ &= (E_{pm}^{(r)}(x))^{1/p}(E_{qn}^{(r)}(y))^{1/q}. \end{aligned}$$

Inequality (13) follows from (12), by replacing m, n by $n/p, n/q$ respectively, and (14) follows from (13) by putting $p = q = 2$. □

Theorem 2.8. *Let $x > 0, 0 < y \le 1, n, m$ are positive integers $m > n$, and m is even. $p > 1, \frac{1}{p}+\frac{1}{q} = 1, 0 < r < 1, \frac{1}{r}+\frac{1}{s} = 1$. Then the following inequalities hold*

$$E_n^{(m)}(xy) \le \left(E_n^{(m)}\left(\frac{rx^p}{p}\right)\right)^{1/p}\left(E_n^{(m)}\left(\frac{sy^q}{q}\right)\right)^{1/q}. \tag{15}$$

In particular

$$(E_n^{(m)}(xy))^2 \le \left(E_n^{(m)}\left(\frac{rx^p}{p}\right)\right)\left(E_n^{(m)}\left(\frac{sy^q}{q}\right)\right). \tag{16}$$

Proof.

$$
\begin{aligned}
E_n^{(m)}(xy) &= \int_1^\infty t^{m-n} e^{-xyt} dt \\
&\geq \int_1^\infty t^{m-n} e^{-\left(\frac{x^p}{p}+\frac{y^q}{q}\right)t} dt \\
&= \int_1^\infty t^{\frac{m}{r}-n} e^{-\frac{x^p}{p}t} t^{\frac{m}{s}-n} e^{-\frac{y^q}{q}t} dt \\
&\geq \left(\int_1^\infty t^{m-rn} e^{-\frac{rx^p}{p}t} dt\right)^{1/r} \left(\int_1^\infty t^{m-sn} e^{-\frac{sy^q}{q}t} dt\right)^{1/s} \\
&= \left(E_n^{(m)}\left(\frac{rx^p}{p}\right)\right)^{1/r} \left(E_n^{(m)}\left(\frac{sy^q}{q}\right)\right)^{1/s}.
\end{aligned}
$$

Inequality (16) follows from (15) by putting $p = q = 2$. □

References

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