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NEW CHARACTERIZATIONS OF WEAK SHARP AND STRICT LOCAL MINIMIZERS IN NONLINEAR PROGRAMMING

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Abstract. We consider the problem of identifying weak sharp local minimizers of order m, an important class of possibly non-isolated local minimizers. A characterization of such minimizers is obtained for a nonlinear programming problem with an abstract set constraint. The results are formulated in terms of certain normal and tangent cones to given sets, and generalized directional derivatives of the objective function. A particular case where the constraint set is given by a system of inequalities is also considered. As a consequence, we obtain a useful characterization of strict local minimizers of order m.

Keywords: weak sharp minimizer; strict local minimizer; Mordukhovich normal cone; contingent cone, interior Ursescu tangent cone; directional derivative.

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1. Introduction

The notion of a weak sharp minimum was introduced by Burke and Ferris in [1]. It is an extension of a strict (or strongly unique [4]) minimum to include the possibility of a non-unique solution set. Weak sharp minima play an important role in the convergence analysis of iterative numerical methods (see Section 4 of [1]). Some results concerning characterizations of such minimizers for constrained optimization problems were derived in [14], with special attention given to weak sharp local minimizers of order two. In [11],

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This paper is dedicated to the memory of my wife Monika.

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a characterization of weak sharp local minimizers of order one for a standard nonlinear programming problem was obtained; the result was stated in terms of Karush–Kuhn–Tucker multipliers. Continuing this line of research, we present here a new characterization of weak sharp minimizers of arbitrary order m for a nonlinear programming problem with an abstract set constraint $x \in C$. As in [11], the optimality conditions are formulated by using a modified Mordukhovich normal cone $N_C(S, x_0)$ (see formula (4) below) and some generalized directional derivative of the objective function. We also obtain a specification of the previous characterization for the case where the constraint set C is defined by a finite number of inequalities. As a consequence of these results, we obtain a characterization of strict local minimizers of order m which improves an earlier theorem proved in [9].

More results on weak sharp minima of scalar-valued functions can be found in [2], [8], [12], [13], [17] and [18].

We end this section by setting some notation and definitions that will be useful in the sequel. Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^n$. For $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, we denote $B(x_0, \varepsilon) := \{x \in \mathbb{R}^n | \, \|x - x_0\| \leq \varepsilon\}$. We also denote by $\mathcal{N}(x_0)$ the collection of all neighborhoods of x_0. For a given subset S of $\mathbb{R}^n$, we define the *distance function* d_S as follows:

$$d_S(x) := \inf\{\|y - x\| \, | y \in S\}.$$

We also denote the *closure* of S by cl$S$, the *boundary* of S by bd$S$, and the *convex hull* of S by co$S$. The *indicator function* of S is the function i_S defined by

$$i_S(x) := \begin{cases} 0, & \text{for } x \in S, \\ +\infty, & \text{for } x \in \mathbb{R}^n \backslash S. \end{cases}$$

Given a locally Lipschitzian function $f : \mathbb{R}^n \to \mathbb{R}$, we denote by $\partial f(x_0)$ the *generalized gradient* of f at x_0 (see [3, p. 27]). It is defined by

$$\partial f(x_0) := \{\xi \in \mathbb{R}^n | f^\circ(x_0; y) \geq \langle \xi, y \rangle , \forall y \in \mathbb{R}^n\},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product, and

$$f^\circ(x_0; y) := \limsup_{(t,x) \to (0^+, x_0)} (f(x + ty) - f(x))/t.$$

We say that a locally Lipschitzian function f is *regular at* x_0 if the usual one-sided directional derivative $f'(x_0; y)$ exists for all y and satisfies the equality

$$f'(x_0; y) = f^\circ(x_0; y).$$

2. A review of tangent cones

In this section, we review several definitions of "tangent" cones to a given subset C of $\mathbb{R}^n$ at a point $x \in C$. These notions will be useful for the derivation of optimality conditions in Sections 3 and 4. We first present the definitions stated in terms of neighborhoods, as in [15], and then also give some equivalent descriptions in terms of sequences.

The four cones of particular importance to us here are the *contingent cone*, defined by

$$K(C, x) := \{y \in \mathbb{R}^n | \forall Y \in \mathcal{N}(y),\, \forall \lambda > 0,\, \exists t \in (0, \lambda),\, \exists y' \in Y,\, x + ty' \in C\},$$

the *Ursescu tangent cone*, defined by

$$k(C, x) := \{y \in \mathbb{R}^n | \forall Y \in \mathcal{N}(y),\, \exists \lambda > 0,\, \forall t \in (0, \lambda),\, \exists y' \in Y,\, x + ty' \in C\},$$

the *interior Ursescu tangent cone*, defined by

$$Ik(C, x) := \{y \in \mathbb{R}^n | \exists Y \in \mathcal{N}(y),\, \exists \lambda > 0,\, \forall t \in (0, \lambda),\, \forall y' \in Y,\, x + ty' \in C\},$$

and the *interior Clarke tangent cone* (called also the *set of hypertangents*), defined by

$$IT(C, x) := \{y \in \mathbb{R}^n | \exists Y \in \mathcal{N}(y),\, \exists X \in \mathcal{N}(x),\, \exists \lambda > 0,$$
$$\forall x' \in X \cap C,\, \forall t \in (0, \lambda),\, \forall y' \in Y,\, x' + ty' \in C\}.$$

Proposition 2.1. *We have:*

(a) $K(C, x) = \{y | \exists \{t_j\} \to 0^+,\, \exists \{y_j\} \to y$ *such that* $x + t_j y_j \in C,\, \forall j\}$,

(b) $K(C, x) = \{y | \exists \{x_j\} \to x,\, \exists \{\lambda_j\} \subset (0, +\infty)$ *with* $x_j \in C$ *and* $y = \lim_{j\to\infty} \lambda_j (x_j - x)\}$,

(c) $k(C, x) = \{y | \forall \{t_j\} \to 0^+,\, \exists \{y_j\} \to y$ *such that* $x + t_j y_j \in C,\, \forall j\}$,

(d) $Ik(C, x) = \{y | \forall \{t_j\} \to 0^+,\, \forall \{y_j\} \to y,\, x + t_j y_j \in C,\, \forall j$ *large enough*$\}$.

Proof. Parts (a) and (b) follow from [5, Theorem 1] ($\mathbf{T}_1 = \mathbf{T}_6 = \mathbf{T}_9$, where one should replace "$\forall\{\lambda_k\}$" by "$\exists\{\lambda_k\}$" in the definition of $\mathbf{T}_1$). Part (c) follows from [5, Theorem 2] ($\mathbf{A}_3 = \mathbf{A}_5$). The proof of part (d) is elementary. □

3. Problems with an abstract set constraint

In this section, we consider the following nonlinear programming problem:

$$\min\{f(x)|x \in C\}, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and C is a nonempty closed subset of $\mathbb{R}^n$.

Let $m \geq 1$ be an integer. Suppose that f is constant on some closed set $S \subset C$, and that $x_0 \in S$. We say that x_0 is a *weak sharp local minimizer of order* m for (1) if there exist $\varepsilon > 0$, $\beta > 0$ such that

$$f(x) - f(x_0) \geq \beta(d_S(x))^m, \quad \forall x \in C \cap B(x_0, \varepsilon). \tag{2}$$

In particular, if $S = \{x_0\}$, we say that x_0 is a *strict local minimizer of order* m for (1). In this case, condition (2) can be replaced by

$$f(x) - f(x_0) \geq \beta \|x - x_0\|^m, \quad \forall x \in C \cap B(x_0, \varepsilon).$$

In order to formulate necessary and sufficient conditions for x_0 to be a weak sharp local minimizer of order m for (1), we now introduce two concepts of normal cones to S at x_0. First, for any $x \in \mathbb{R}^n$, call

$$P(S, x) := \{w \in S| \|x - w\| = d_S(x)\}.$$

Now, let $x_0 \in S$. The *normal cone* to S at x_0 (in the sense of Mordukhovich) is defined by

$$N(S, x_0) := \{y \in \mathbb{R}^n|\exists\{y_j\} \to y,\, \{x_j\} \to x_0,\, \{t_j\} \subset (0, +\infty),\, \{s_j\} \subset \mathbb{R}^n \text{ with } s_j \in P(S, x_j) \text{ and } y_j = (x_j - s_j)/t_j\}. \tag{3}$$

We also introduce a variation of the normal cone that takes the set C into account. The *normal cone* to S at x_0 relative to C is defined by

$$N_C(S,x_0) := \{y \in \mathbb{R}^n | \exists \{y_j\} \to y,\ \{x_j\} \to x_0,\ \{t_j\} \subset (0,+\infty),\ \{s_j\} \subset \mathbb{R}^n \text{ with } x_j \in C,\ s_j \in P(S,x_j) \text{ and } y_j = (x_j - s_j)/t_j\}. \tag{4}$$

Comparing (3) and (4), we see that

$$N_C(S,x_0) \subset N(S,x_0). \tag{5}$$

The following theorem is an extension of Theorem 2.2 in [14] to the case of constrained optimization problem (1).

Theorem 3.1. *Let S, C be nonempty closed subsets of $\mathbb{R}^n$, such that $S \subset C$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be constant on S, and let $x_0 \in S$. Then, for each $m \geq 1$, the following conditions are equivalent:*

(a) *x_0 is a weak sharp local minimizer of order m for* (1)*, that is, condition* (2) *is satisfied for some $\varepsilon > 0$, $\beta > 0$;*

(b) *for all $y \in N_C(S,x_0)$ with $\|y\| = 1$, for all sequences $\{x_j\} \to x_0$, $\{s_j\}$ with $x_j \in C$, $s_j \in P(S,x_j)$, $\{(x_j - s_j)/\|x_j - s_j\|\} \to y$, and*

$$\liminf_{j\to\infty} (f(x_j) - f(s_j))/\|x_j - s_j\|^{m-1} \leq 0, \tag{6}$$

we have

$$\liminf_{j\to\infty} (f(x_j) - f(s_j))/\|x_j - s_j\|^{m} > 0; \tag{7}$$

(c) *condition* (7) *holds under the assumptions stated in part* (b) *except for* (6).

Proof. (a) $\Longrightarrow$ (c): Suppose that (a) holds. Let $y \in N_C(S,x_0)$ with $\|y\| = 1$, and let $\{x_j\} \to x_0$ and $\{s_j\}$ be such that $x_j \in C$, $s_j \in P(S,x_j)$ and

$$\{(x_j - s_j)/\|x_j - s_j\|\} \to y.$$

Since $s_j \in S$ and $x_j \in C$, it follows by (a) that there exists $\beta > 0$ such that, for j large enough,

$$f(x_j) - f(s_j) = f(x_j) - f(x_0) \geq \beta(d_S(x_j))^m = \beta\|x_j - s_j\|^m.$$

Thus

$$\liminf_{j\to\infty}(f(x_j)-f(s_j))/\|x_j-s_j\|^m \geq \beta > 0,$$

and (c) holds.

(c) $\Longrightarrow$ (b) is obvious.

(b) $\Longrightarrow$ (a) (by contraposition): Suppose that x_0 is not a weak sharp local minimizer of order m for (1). Then there exists a sequence $\{x_j\} \to x_0$ such that $x_j \in C$ and

$$f(x_j)-f(x_0) < (d_S(x_j))^m/j. \tag{8}$$

For each j, let $s_j \in P(S,x_j)$. Inequality (8) implies that $x_j \notin S$, and so $x_j \neq s_j$. Taking a subsequence if necessary, we may assume without loss of generality that the sequence $\{(x_j-s_j)/\|x_j-s_j\|\}$ converges to some y. Then $y \in N_C(S,x_0)$, $\|y\|=1$, and by (8),

$$f(x_j)-f(s_j) = f(x_j)-f(x_0) < (d_S(x_j))^m/j = \|x_j-s_j\|^m/j.$$

Hence, using the estimate $\|x_j-s_j\| \leq \|x_j-x_0\| \to 0$, we find that (6) holds, and

$$\liminf_{j\to\infty}(f(x_j)-f(s_j))/\|x_j-s_j\|^m \leq 0.$$

This contradicts (b), and the proof is complete. □

Condition (b) of Theorem 3.1 gives a general characterization of weak sharp local minimizers of order m for (1). However, it is rather difficult to apply it in practice. Therefore, we need some easier conditions stated in terms of certain directional derivatives of f. Following [16], we will use the notation

$$d^m f^K(x;y) := \liminf_{(t,v)\to(0^+,y)} (f(x+tv)-f(x))/t^m.$$

For $m=1$, we will write f^K instead of $d^1 f^K$. The relation between $d^m f^K$ and the usual m-order derivatives of f (if they exist) is discussed in [9]. We will also use the following modification of $d^m f^K$ introduced in [14]. Let A be a nonempty subset of $\mathbb{R}^n$, and let $x \in \operatorname{bd} A$. For $y \in \mathbb{R}^n$, define

$$d^m f^K_A(x;y) := \liminf_{\substack{\operatorname{bd} A \ni s \to x \\ (t,v)\to(0^+,y)}} (f(s+tv)-f(s))/t^m.$$

Now, consider again two closed sets S, C such that $S \subset C$. For $x_0 \in S$, we define

$$S(x_0) := \{x_0\} \cup \bigcup_{x \in C \setminus S} P(S, x).$$

Using these concepts, we can prove a sufficient condition for weak sharp local minimality of order m.

Theorem 3.2. *Let S, C be nonempty closed subsets of $\mathbb{R}^n$, such that $S \subset C$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be constant on S, and let $x_0 \in S$. If $m \geq 1$ and*

$$d^m f^K_{S(x_0)}(x_0; y) > 0, \quad \forall y \in N_C(S, x_0) \setminus \{0\}, \tag{9}$$

then x_0 is a weak sharp local minimizer of order m for (1).

Proof. Let $y \in N_C(S, x_0)$ with $\|y\| = 1$. In order to verify condition (c) of Theorem 3.1, let us take any sequences $\{x_j\} \to x_0$, $\{s_j\}$ such that $x_j \in C$, $s_j \in P(S, x_j)$ and $\{(x_j - s_j)/\|x_j - s_j\|\} \to y$. Then $x_j \notin S$, and so $s_j \in \mathrm{bd}S(x_0)$. Define $t_j := \|x_j - s_j\|$ and $y_j := (x_j - s_j)/t_j$. By the definition of $P(S, x_j)$, we have $\|x_j - s_j\| \leq \|x_j - x_0\|$, which implies $t_j \to 0^+$ and $s_j \to x_0$. Now by (9), it follows that

$$\begin{aligned} \liminf_{j\to\infty} (f(x_j) - f(s_j))/\|x_j - s_j\|^m &= \liminf_{j\to\infty} (f(s_j + t_j y_j) - f(s_j))/t_j^m \\ &\geq d^m f^K_{S(x_0)}(x_0; y) > 0. \end{aligned}$$

Hence x_0 is a weak sharp local minimizer of order one for (1) by Theorem 3.1. □

A general necessary condition for a weak sharp local minimum of order m for (1) has been obtained in [16, Theorem 4.1]. However, it involves the indicator function i_C of the constraint set C, which is very inconvenient from the practical point of view. Below we apply Ward's theorem to derive other forms of necessary conditions which do not contain the indicator function. One of them will be used later to obtain a characterization of weak sharp local minima of order m.

Theorem 3.3. *Let S, C be nonempty closed subsets of $\mathbb{R}^n$, such that $S \subset C$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be constant on S, and let $x_0 \in \mathrm{bd}S$ and $m \geq 1$. Suppose that x_0 is a weak sharp local minimizer of order m for* (1). *Then:*

(a) *there exists* $\beta > 0$ *such that*

$$d^m f^K(x_0; y) \geq \beta(d_{K(S,x_0)}(y))^m, \quad \forall y \in Ik(C, x_0); \tag{10}$$

(b) *we have*

$$d^m f^K(x_0; y) > 0, \quad \forall y \in Ik(C, x_0)\backslash K(S, x_0);$$

(c) *if the restriction of* $d^m f^K(x_0; \cdot)$ *to* $\mathrm{cl}Ik(C, x_0)$ *is finite and continuous, we have*

$$d^m f^K(x_0; y) > 0, \quad \forall y \in (\mathrm{cl}Ik(C, x_0))\backslash K(S, x_0). \tag{11}$$

Proof. (a) Suppose that (2) holds for some $\varepsilon > 0$, $\beta > 0$. We first apply [16, Theorem 4.1] to obtain

$$d^m(f + i_C)^K(x_0; y) \geq \beta(d_{K(S,x_0)}(y))^m, \quad \forall y \in \mathbb{R}^n. \tag{12}$$

(Note that Ward uses a slightly different definition of a weak sharp local minimizer of order m, where the set S is replaced by $S \cap B(x_0, \varepsilon)$. However, it is easy to see that the same proof can be repeated for our definition (2).)

Now, let $y \in Ik(C, x_0)$. We want to transform the left-hand side of (12) by using [16, Corollary 2.1(ii)]. To that end, we must verify the assumption $d^m f^K(x_0; y) > -\infty$ (in the formulation of this assumption in [16, Corollary 2.1], the symbol d^m is omitted by mistake). By the definition of $d^m f^K$, there exist sequences $\{t_j\} \to 0^+$ and $\{y_j\} \to y$ such that

$$d^m f^K(x_0; y) = \lim_{j\to\infty} (f(x_0 + t_j y_j) - f(x_0))/t_j^m. \tag{13}$$

It follows from Proposition 2.1(d) that $x_0 + t_j y_j \in C \cap B(x_0, \varepsilon)$ for j large enough. Hence, using (2), we obtain for such j,

$$f(x_0 + t_j y_j) - f(x_0) \geq \beta(d_S(x_0 + t_j y_j))^m \geq 0. \tag{14}$$

Conditions (13) and (14) imply $d^m f^K(x_0; y) \geq 0 > -\infty$. Therefore, we can apply [16, Corollary 2.1(ii)] and the equality

$$d^m i_C^{Ik}(x_0; \cdot) = i_{Ik(C,x_0)}(\cdot)$$

(see [16, p. 556]) to obtain

$$d^m(f+i_C)^K(x_0;y) \le d^m f^K(x_0;y) + i_{Ik(C,x_0)}(y) = d^m f^K(x_0;y). \tag{15}$$

The desired inequality (10) follows from (12) and (15).

(b) Let $y \in Ik(C,x_0)\backslash K(S,x_0)$. Since $y \notin K(S,x_0)$ and $K(S,x_0)$ is closed, we have $d_{K(S,x_0)}(y) > 0$, and so $d^m f^K(x_0;y) > 0$ by (10).

(c) Let $y \in (\mathrm{cl}Ik(C,x_0))\backslash K(S,x_0)$. There exists a sequence $\{y_j\} \subset Ik(C,x_0)$ such that $\{y_j\} \to y$. As in part (b), we have $d_{K(S,x_0)}(y) > 0$. By part (a), there exists $\beta > 0$ such that

$$d^m f^K(x_0;y_j) \ge \beta(d_{K(S,x_0)}(y_j))^m, \quad \text{for all } j.$$

Taking the limits of both sides as $j \to \infty$, and using the continuity of $d^m f^K(x_0;\cdot)$ and $d_{K(S,x_0)}(\cdot)$, we obtain

$$d^m f^K(x_0;y) \ge \beta(d_{K(S,x_0)}(y))^m > 0.$$

□

By comparing Theorems 3.2 and 3.3, we can determine assumptions under which condition (11) characterizes weak sharp local minima of order m for (1). More precisely, we can prove the following result.

Theorem 3.4. *Let S, C be nonempty closed subsets of $\mathbb{R}^n$, such that $S \subset C$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be constant on S, and let $x_0 \in \mathrm{bd}S$ and $m \ge 1$. Suppose that:*

(i) *$N_C(S,x_0)\backslash\{0\} \subset (\mathrm{cl}Ik(C,x_0))\backslash K(S,x_0)$;*

(ii) *$d^m f^K(x_0;y) = d^m f^K_{S(x_0)}(x_0;y)$, for all $y \in N_C(S,x_0)$;*

(iii) *the restriction of $d^m f^K(x_0;\cdot)$ to $\mathrm{cl}Ik(C,x_0)$ is finite and continuous.*

Then the following conditions are equivalent for $m > 1$:

(a) *x_0 is a weak sharp local minimizer of order m for (1);*

(b) *$d^m f^K(x_0;y) > 0$, for all $y \in (\mathrm{cl}Ik(C,x_0))\backslash K(S,x_0)$;*

(c) *$d^m f^K(x_0;y) > 0$, for all $y \in (\mathrm{cl}Ik(C,x_0))\backslash K(S,x_0)$ such that $f^K(x_0;y) \le 0$.*

For $m = 1$, conditions (a) *and* (b) *are equivalent (condition* (c) *obviously does not make sense).*

Proof. We give the proof for $m > 1$. The easier proof for $m = 1$ is omitted.

(a) $\Longrightarrow$ (b): This implication follows from Theorem 3.3(c) and assumption (iii).

(b) $\Longrightarrow$ (c) is obvious.

(c) $\Longrightarrow$ (a): Suppose that (c) holds. In order to verify condition (9), let us take any $y \in N_C(S, x_0)\backslash\{0\}$. By assumption (i), $y \in (\mathrm{cl}Ik(C, x_0))\backslash K(S, x_0)$. Now, if $f^K(x_0; y) \leq 0$, then

$$d^m f^K_{S(x_0)}(x_0; y) = d^m f^K(x_0; y) > 0 \tag{16}$$

by assumption (ii) and condition (c). Otherwise, we have $f^K(x_0; y) > 0$, and so

$$\begin{aligned} d^m f^K(x_0; y) &= \liminf_{(t,v)\to(0^+,y)} (f(x + tv) - f(x))/t^m \\ &\geq \left(\liminf_{(t,v)\to(0^+,y)} (f(x + tv) - f(x))/t\right)\left(\liminf_{t\to0^+} 1/t^{m-1}\right) \\ &= f^K(x_0; y) \cdot (+\infty) = +\infty. \end{aligned}$$

Hence, using again assumption (ii), we find that (16) is also valid. Applying Theorem 3.2, we conclude that x_0 is a weak sharp local minimizer of order m for (1). □

In [14, Theorem 2.7], a characterization of weak sharp local minimizers of order m for unconstrained problems (i.e., with $C = \mathbb{R}^n$) was obtained under the assumption $K(S, x_0) \cap N(S, x_0) = \{0\}$. This condition holds, in particular, when the set S is convex (see [14, Remark 1(a)]). We now prove an analogue of that characterization for constrained optimization problems.

Theorem 3.5. *Let S, C be nonempty closed subsets of $\mathbb{R}^n$, such that $S \subset C$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be constant on S, and let $x_0 \in \mathrm{bd}S$ and $m \geq 1$. Suppose that:*

(i) *$K(S, x_0) \cap N(S, x_0) = \{0\}$;*

(ii) *$N_C(S, x_0) \subset \mathrm{cl}Ik(C, x_0)$;*

(iii) *$d^m f^K(x_0; y) = d^m f^K_{S(x_0)}(x_0; y)$, for all $y \in N_C(S, x_0)$;*

(iv) *the restriction of $d^m f^K(x_0; \cdot)$ to $\mathrm{cl}Ik(C, x_0)$ is finite and continuous.*

Then the following conditions are equivalent for $m > 1$:

(a) *x_0 is a weak sharp local minimizer of order m for (1);*

(b) $d^m f^K(x_0;y) > 0$, *for all* $y \in (N(S,x_0) \cap \mathrm{cl}Ik(C,x_0))\backslash\{0\}$*;*

(c) $d^m f^K(x_0;y) > 0$, *for all* $y \in (N(S,x_0) \cap \mathrm{cl}Ik(C,x_0))\backslash\{0\}$ *such that* $f^K(x_0;y) \le 0$.

For $m = 1$, *conditions* (a) *and* (b) *are equivalent.*

Proof. ($m > 1$). (a) $\Longrightarrow$ (b): It follows from assumption (i) that

$$(N(S,x_0) \cap \mathrm{cl}Ik(C,x_0))\backslash\{0\} \subset (\mathrm{cl}Ik(C,x_0))\backslash K(S,x_0).$$

Observe also that assumptions (i) and (ii) imply assumption (i) of Theorem 3.4. Hence, the desired implication is obtained from Theorem 3.4 ((a) $\Longrightarrow$ (b)).

(b) $\Longrightarrow$ (c) is obvious.

(c) $\Longrightarrow$ (a): Suppose that (c) holds. In order to verify condition (9), let us take any $y \in N_C(S,x_0)\backslash\{0\}$. By inclusion (5) and assumption (ii), we have $y \in (N(S,x_0) \cap \mathrm{cl}Ik(C,x_0))\backslash\{0\}$. Hence, by assumption (i), $y \notin K(S,x_0)$. The rest of the proof is the same as in Theorem 3.4 ((c) $\Longrightarrow$ (a)). □

It should be noted that further investigation is required to obtain verifiable criteria for assumptions (i)–(iii) of Theorem 3.5 to hold. This is difficult because the set S is usually not given explicitly. Fortunately, these assumptions can be considerably simplified for the particular but important case of strict local minimizers of order m, which is done in Corollary 3.1 below. First however, we present an example illustrating Theorem 3.5.

Example 3.1. *Let* $m = 2$, *and let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be given by*

$$f(x_1,x_2) := \begin{cases} x_1^2 + x_2, & \text{if } x_2 \ge 0, \\ x_1^2, & \text{if } x_2 < 0. \end{cases}$$

Define $C := \{(x_1,x_2) | x_2 \le 0\}$ *and* $S := \{(x_1,x_2) | x_1 = 0,\ x_2 \le 0\}$. *Observe that* f *is constant on* S, *and the point* $(0,0)$ *is a weak sharp local (and also global) minimizer of order* 2 *for problem* (1). *We can easily calculate:*

$K(S,(0,0)) = S$,

$N(S,(0,0)) = \{(y_1,y_2) | y_2 \ge 0\}$,

$N_C(S,(0,0)) = \{(y_1,y_2) | y_2 = 0\}$,

$\mathrm{cl}Ik(C,(0,0)) = C$.

Hence, assumptions (i) *and* (ii) *of Theorem 3.5 are satisfied. We can also verify that*

$$d^2 f^K((0,0);(y_1,y_2)) = \begin{cases} y_1^2, & \text{if } y_2 \le 0, \\ +\infty, & \text{if } y_2 > 0. \end{cases}$$

This shows that $d^2 f^K((0,0);\cdot)$ *is continuous when restricted to* C*, but not continuous on the whole space* $\mathbb{R}^2$*. Thus, assumption* (iv) *is satisfied. Assumption* (iii) *can also be verified by a direct computation (we omit the details). Since*

$$(N(S,(0,0)) \cap \mathrm{cl}Ik(C,(0,0)))\backslash\{(0,0)\} = \{(y_1,y_2)|y_1 \ne 0,\ y_2 = 0\},$$

we see that condition (b) *of Theorem 3.5 holds. Condition* (c) *is the same as* (b) *since* $f^K((0,0);(y_1,0)) = 0$ *for all* y_1.

Corollary 3.1. *Let* C *be a nonempty closed subset of* $\mathbb{R}^n$*, let* $f : \mathbb{R}^n \to \mathbb{R}$*, and let* $x_0 \in C$ *and* $m \ge 1$*. Suppose that:*

(i) $K(C,x_0) \subset \mathrm{cl}Ik(C,x_0)$*;*

(ii) *the restriction of* $d^m f^K(x_0;\cdot)$ *to* $\mathrm{cl}Ik(C,x_0)$ *is finite and continuous.*

Then the following conditions are equivalent for $m > 1$*:*

(a) x_0 *is a strict local minimizer of order* m *for* (1)*;*

(b) $d^m f^K(x_0;y) > 0$*, for all* $y \in K(C,x_0)\backslash\{0\}$*;*

(c) $d^m f^K(x_0;y) > 0$*, for all* $y \in K(C,x_0)\backslash\{0\}$ *such that* $f^K(x_0;y) \le 0$*.*

For $m = 1$*, conditions* (a) *and* (b) *are equivalent.*

Proof. Let $S = \{x_0\}$, then $K(S,x_0) = \{0\}$ and $N(S,x_0) = \mathbb{R}^n$, so that assumption (i) of Theorem 3.5 is satisfied. Furthermore, it follows from (4) and Proposition 2.1(b) that $N_C(S,x_0) = K(C,x_0)$, which means that (i) is equivalent to (ii) of Theorem 3.5. Observe also that (iii) of Theorem 3.5 holds trivially. Finally, let us note that the inclusion $\mathrm{cl}Ik(C,x_0) \subset K(C,x_0)$ is an easy consequence of the definitions of the respective cones and the closedness of $K(C,x_0)$. Therefore, (i) gives the equality

$$K(C,x_0) = \mathrm{cl}Ik(C,x_0), \tag{17}$$

and consequently, $N(S, x_0) \cap \mathrm{cl}Ik(C, x_0) = K(C, x_0)$. The desired conclusion now follows from Theorem 3.5. □

4. Problems with inequality-type constraints

We now assume that the constraint set C in problem (1) is given by

$$C := \{x \in \mathbb{R}^n | g_1(x) \leq 0, ..., g_l(x) \leq 0\}, \tag{18}$$

where $g_1, ..., g_l$ are locally Lipschitzian functions from $\mathbb{R}^n$ to $\mathbb{R}$.

First, we will discuss a constraint qualification which ensures the fulfillment of condition (17).

Lemma 4.1. *Suppose that $x_0 \in C$ and the functions $g_1, ..., g_l$ are regular at x_0. Define*

$$I(x_0) := \{i \in \{1, ..., l\} | g_i(x_0) = 0\}.$$

If

$$0 \notin \mathrm{co} \bigcup_{i \in I(x_0)} \partial g_i(x_0), \tag{19}$$

then

$$K(C, x_0) = k(C, x_0) = \mathrm{cl}Ik(C, x_0) = C(x_0), \tag{20}$$

where $C(x_0)$ is defined by

$$C(x_0) := \{y \in X | g_i'(x_0; y) \leq 0, \forall i \in I(x_0)\}.$$

Proof. Let $g := \max\{g_i | i \in I(x_0)\}$. Then g is locally Lipschitzian, and by [7, Theorem 3.2.13], we have

$$\partial g(x_0) = \mathrm{co} \bigcup_{i \in I(x_0)} \partial g_i(x_0). \tag{21}$$

Define $C_g := \{x \in \mathbb{R}^n | g(x) \leq 0\}$. By the continuity of g_i for $i \notin I(x_0)$, there exists $X \in \mathcal{N}(x_0)$ such that $g_i(x) < 0$ for all $i \notin I(x_0)$ and $x \in X$. Therefore, we have $C_g \cap X = C \cap X$, and consequently,

$$IT(C_g, x_0) = IT(C, x_0). \tag{22}$$

Since $g(x_0) = 0$ and $0 \notin \partial g(x_0)$ (by (19) and (21)), we may apply [3, Theorem 2.9.10] to deduce that C_g admits a hypertangent at x_0, which means that $IT(C_g, x_0) \neq \emptyset$. Hence, $IT(C, x_0) \neq \emptyset$ by (22). This last condition is required in [15, Corollary 2.8] to prove that

$$k(C, x_0) = \mathrm{cl} Ik(C, x_0) \tag{23}$$

(we use here the case $A = k$ of Ward's result). Finally, we observe that, for the set C defined by (18), condition (19) is equivalent to assumption (A1) in [10]. Hence, we may apply [10, Theorem 2] to obtain

$$K(C, x_0) = k(C, x_0) = C(x_0).$$

This together with (23) gives (20). □

Combining Theorem 3.5 and Lemma 4.1, we get the following.

Theorem 4.1. *Consider problem* (1) *in which the constraint set* C *is defined by* (18), *where the functions* $g_1, ..., g_l$ *are locally Lipschitzian and regular at* $x_0 \in C$. *Let* S *be a nonempty closed subset of* C *such that* $x_0 \in \mathrm{bd} S$ *and* $f : \mathbb{R}^n \to \mathbb{R}$ *is constant on* S. *Suppose that:*

(i) *condition* (19) *is fulfilled;*

(ii) $K(S, x_0) \cap N(S, x_0) = \{0\}$;

(iii) $N_C(S, x_0) \subset C(x_0)$;

(iv) $d^m f^K(x_0; y) = d^m f^K_{S(x_0)}(x_0; y)$, *for all* $y \in N_C(S, x_0)$;

(v) *the restriction of* $d^m f^K(x_0; \cdot)$ *to* $C(x_0)$ *is finite and continuous.*

Then the following conditions are equivalent for $m > 1$:

(a) x_0 *is a weak sharp local minimizer of order* m *for* (1);

(b) $d^m f^K(x_0; y) > 0$, *for all* $y \in (N(S, x_0) \cap C(x_0)) \backslash \{0\}$;

(c) $d^m f^K(x_0; y) > 0$, *for all* $y \in (N(S, x_0) \cap C(x_0)) \backslash \{0\}$ *such that* $f^K(x_0; y) \leq 0$.

For $m = 1$, *conditions* (a) *and* (b) *are equivalent.*

The following corollary follows immediately from Corollary 3.1 and Lemma 4.1.

Corollary 4.1. *Consider problem* (1) *in which the constraint set* C *is defined by* (18), *where the functions* $g_1, ..., g_l$ *are locally Lipschitzian and regular at* $x_0 \in C$. *Let* $m \geq 1$, *and let the restriction of* $d^m f^K(x_0; \cdot)$ *to* $C(x_0)$ *be finite and continuous. Suppose also that* (19) *holds. Then the following conditions are equivalent for* $m > 1$*:*

(a) x_0 *is a strict local minimizer of order* m *for* (1)*;*

(b) $d^m f^K(x_0; y) > 0$, *for all* $y \in C(x_0)\backslash\{0\}$*;*

(c) $d^m f^K(x_0; y) > 0$, *for all* $y \in C(x_0)\backslash\{0\}$ *such that* $f^K(x_0; y) \leq 0$.

For $m = 1$, *conditions* (a) *and* (b) *are equivalent.*

For the particular case considered in this section, Corollary 4.1 improves an earlier characterization of strict local minimizers of order m presented in [9, Theorem 2.1]. This improvement is twofold. Firstly, the indicator function of C is no longer involved in optimality conditions (b) and (c). Secondly, the contingent cone $K(C, x_0)$ is replaced by the set $C(x_0)$ which is described explicitly in terms of the functions $g_1, ..., g_l$. These modifications facilitate the practical use of optimality conditions. Other conditions for strict local minima, involving the Lagrange function, have been developed in [6].

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